Scattering for the focusing $\dot{H}^{1/2}$-critical Hartree equation with radial data

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Abstract

We investigate the focusing $\dot{H}^{1/2}$-critical nonlinear Schrödinger equation (NLS) of Hartree type $i\partial_t u + \Delta u = -(|\cdot|^{-3} * |u|^2)u$ with $\dot{H}^{1/2}$ radial data in dimension $d = 5$. It is proved that if the maximal life-span solution obeys $\sup_t \big\||\nabla|^{\frac{1}{2}} u\big\|_2 < \frac{\sqrt{6}}{3}\big\||\nabla|^{\frac{1}{2}} Q\big\|_2$, where Q is the positive radial solution to the elliptic equation with nonlocal operator (1.4) which corresponds to a new variational structure. Then the solution is global and scatters.

Key Words: Hartree equation, scattering, profiles decomposition, almost periodic solution, concentration compactness
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1 Introduction

Consider the Cauchy problem for the $\dot{H}^{1/2}$-critical Hartree equation

$$i\partial_t u + \Delta u = F(u) \tag{1.1}$$

in $\mathbb{R}^5$, where $F(u) = -(|\cdot|^{-3} * |u|^2)u$, u is a complex-valued function defined on some spacetime slab $I \times \mathbb{R}^5$. The Hartree equation arises in the study of boson stars and other physical phenomena, see, for instance, [25].

The term $\dot{H}^{1/2}$-critical means that the scaling

$$u_\lambda(t, x) = \lambda^{-2} u(\lambda^{-2} t, \lambda^{-1} x) \tag{1.2}$$

leaves both the equation and the initial data of $\dot{H}^{1/2}_x$- norm invariant. By a function $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ is a *solution* to (1.1), it means that $u \in C^0_t \dot{H}^{1/2}_x(K \times \mathbb{R}^5) \cap L^3_t L^{15/4}_x(K \times \mathbb{R}^5)$ for any compact $K \subset I$, and u obeys the Duhamel formula

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^{t} e^{i(t-t')\Delta} F(u(t'))\, \mathrm{d}t'$$

for all t, $t_0 \in I$. We call I the *life-span* of u. If I can not be extended strictly larger, we say I is the *maximal life-span* of u, and u is a *maximal life-span solution.* If $I = \mathbb{R}$, then u is global.

Definition 1.1 (Blow up). *Let $u : I \times \mathbb{R}^d \mapsto \mathbb{C}$ be a solution to (1.1). Say u blows up forward in time if there exists $t_1 \in I$ such that*

$$\|u\|_{L_t^3 L_x^{15/4}([t_1,\, \sup I)\times\mathbb{R}^5)} = \infty\,;$$

and u blows up backward in time if there exists t_1 such that

$$\|u\|_{L_t^3 L_x^{15/4}((\inf I,\, t_1]\times\mathbb{R}^5)} = \infty.$$

Throughout the paper, we write

$$\|u\|_{S(I)} := \|u\|_{L_t^3 L_x^{15/4}(I\times\mathbb{R}^5)}, \quad \|u\|_{X(I)} := \big\||\nabla|^{\frac{1}{2}} u\big\|_{L_t^3 L_x^{30/11}(I\times\mathbb{R}^5)}.$$

The local theory for (1.1) was established by Cazenave and Weissler [3], [4]. Using a fixed point argument together with Strichartz's estimates in the framework of Besov spaces, they constructed local in time solution for arbitrary initial data. However, due to the critical nature of the equation, the existence time depends on the profile of the initial data and not merely on its $\dot{H}_x^{1/2}$-norm. They also proved the global existence for small data.

Theorem 1.1 (Local theory, [3], [4]). *Let $u_0 \in \dot{H}_x^{1/2}(\mathbb{R}^5)$, $t_0 \in \mathbb{R}$, there exists a unique maximal life-span solution $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ to* (1.1) *with initial data $u(t_0) = u_0$. This solution also has the following properties:*

- *(Local existence) I is an open neighborhood of t_0.*
- *(Blow up criterion) If* $\sup I$ *is finite, then u blows up forward in time; if* $\inf I$ *is finite, then u blows up backward in time.*
- *(Scattering) If* $\sup I = +\infty$, *and u does not blow up forward in time, then u scatters forward in time, that is, there exists a unique $u_+ \in \dot{H}_x^{1/2}(\mathbb{R}^5)$ such that*

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}u_+\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} = 0. \tag{1.3}$$

  Conversely, given $u_+ \in \dot{H}_x^{1/2}(\mathbb{R}^5)$, there exists a unique solution to (1.1) *in a neighborhood of infinity such that* (1.3) *holds.*

- *(Small data scattering) If $\big\||\nabla|^{\frac{1}{2}} u_0\big\|_2$ is sufficiently small, then u scatters in both time directions. Indeed, $\|u\|_{S(\mathbb{R})} \lesssim \big\||\nabla|^{\frac{1}{2}} u_0\big\|_2$.*
- *(Radial symmetry) If u_0 is radially symmetric, then u remains radially symmetric for all time.*

From Theorem 1.1, a solution to (1.1) with small data must be scattering. However, the result is unknown for arbitrary data, even in the defocusing case. In [10], Kenig and Merle proved for the defocusing cubic NLS that the solution is global and scatters if it remains uniformly bounded in $\dot{H}_x^{1/2}$ on its maximal life-span. The assumption that the solution is uniformly bounded in $\dot{H}_x^{1/2}$ plays a role of the missing conservation law. The argument presented there applies to the corresponding defocusing Hartree equation without difficulty. As to the focusing case, there has been no result on the line of scattering, neither NLS nor of Hartree type. Our primary goal in this paper is to establish scattering result for the focusing Hartree equation, and we believe that the argument can be adapted to the focusing NLS.

For the Cauchy problem of (1.1), there is a stationary solution $e^{it}\bar{Q}$ that is global but blows up both forward and backward. Here $\bar{Q}$ is the unique positive radial Schwartz solution to

$$\Delta\bar{Q} + (|\cdot|^{-3} * |\bar{Q}|^2)\bar{Q} = \bar{Q}.$$

In the focusing energy/mass critical case, the corresponding stationary solution/ground state play the role of an obstruction to the global well-posedness and scattering. Indeed, the global existence follows so long as the kinetic energy/mass of the initial data is strictly less than that of the stationary solution/ground state. In [17], Li-Zhang classify the minimal blowup solutions of the focusing mass-critical Hartree equation. However, wether the solution u to (1.1) on its maximal life-span with $\|u\|_{L_t^\infty \dot{H}_x^{1/2}} < \|\bar{Q}\|_{\dot{H}_x^{1/2}}$ implies global existence is still open. In this paper we will introduce a new elliptic equation:

$$\Delta Q + (|\cdot|^{-3} * |Q|^2)Q = (-\Delta)^{1/2}Q, \tag{1.4}$$

which corresponds to a new variational structure, and prove that if the solution u to (1.1) satisfies $\|u\|_{L_t^\infty \dot{H}_x^{1/2}} < \frac{\sqrt{6}}{3}\|Q\|_{\dot{H}_x^{1/2}}$, then the solution is global and scatters.

Solutions to critical NLS and of Hartree type have been intensively studied, especially those of energy critical equations. Scattering results for the defocusing energy-critical equations have been completely established. These were accomplished by Bourgain [2], Grillakis [7], Tao [23], Colliander-Keel-Staffilani-Takaoka-Tao [5], Ryckman-Visan [24], and Visan [29], Miao-Xu-Zhao [21]. As will be discussed later, the focusing energy-critical NLS theory has also been well established by Kenig-Merle and Killip-Visan, except for dimensions 3 and 4. For the focusing Hartree, it was proved by Li-Miao-Zhang [16], and Miao-Xu-Zhao[23].

Another kind of critical NLS and of Hartree type which receives lots of attention is the mass-critical one. Results in earlier work which is devoted to global well-posedness were usually obtained under the assumption of the H_x^1 initial data. See, e.g., [3], [30]. In [30], Weinstein first observed the role of the ground state for the focusing mass-critical NLS despite finite energy. As far as L_x^2 initial data is concerned, Tao-Visan-Zhang [27] proved the scattering results for the defocusing case for large spherically symmetric data in dimensions three and higher. More recent and nice work on scattering results for L_x^2 data were done by Killip-Tao-Visan [13], Killip-Visan-Zhang [15], and Miao-Xu-Zhao [22] with spherical symmetry assumption.

The recent progress in studying those equations is due to a new and highly efficient approach based on a concentration compactness idea to provide a linear profile decomposition. This approach arises from investigating the defect of compactness for the Strichatz estimates. Based on a refined Sobolev inequality, Kerrani [12] obtained a linear profile decomposition for solutions of free NLS with H_x^1 data. It was Kenig and Merle who first introduced Kerrani's linear profile decomposition to obtain scattering results. They treated the focusing energy-critical NLS in dimensions 3, 4, 5 in [9]. Using the same decomposition, Killip and Visan [14] dealt with the focusing energy-critical NLS in dimensions five and higher without radial assumption. Using the decomposition of [19], Tao-Visan-Zhang [28] made a reduction for failure of scattering. And by combining the reduction with an in/out decomposition technique, [13], [15] settled the scattering problem for the mass-critical NLS with spherically symmetric data.

A linear profile decomposition for general $\dot{H}^s$ data was proved by Shao [26]. Unlike Kerrani's approach which is based on a refined Sobolev inequality, Shao took advantage of the existing L_x^2 linear profile decomposition and the Galilean transform, and managed to eliminate the frequency parameter from the decomposition. In this paper, we will use Shao's linear profile decomposition, and our main result is:

Theorem 1.2. *Let $u_0 \in \dot{H}_x^{1/2}(\mathbb{R}^5)$, radially symmetric, $t_0 \in \mathbb{R}$, I is a time interval containing t_0. Let $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ be a maximal life-span solution to* (1.1)*. Assume $\sup_{t\in I} \big\||\nabla|^{\frac{1}{2}} u(t)\big\|_2 < \frac{\sqrt{6}}{3}\big\||\nabla|^{\frac{1}{2}} Q\big\|_2$. Then u is global and scatters with*

$$\|u\|^3_{L_t^3 L_x^{15/4}(\mathbb{R}\times\mathbb{R}^5)} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^5} |u(t,x)|^{15/4}\,\mathrm{d}x\right)^{4/5} \mathrm{d}t < \infty.$$

Remark 1.1. *It is an interesting problem to describe the correspondence between Q and $\bar{Q}$, and thus leading to some investigation with the gap. It is also an interesting problem that wether the solution blows up so long as $\sup\limits_{t\in I} \big\||\nabla|^{\frac{1}{2}} u(t)\big\|_2 \geq \frac{\sqrt{6}}{3}\big\||\nabla|^{\frac{1}{2}} Q\big\|_2$.*

The concentration compactness argument reduces matters to the study of almost periodic solutions modulo symmetries.

Definition 1.2 (Almost periodic modulo scaling)**.** *Let u be a solution to* (1.1) *with maximal life-span I. Say u is almost periodic modulo scaling if there exist functions $N : I \mapsto \mathbb{R}^+$, $C : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that for all $\eta > 0$, $t \in I$*

$$\int_{|x|\geq C(\eta)/N(t)} \left||\nabla|^{\frac{1}{2}} u(t,x)\right|^2 \mathrm{d}x \leq \eta$$

and

$$\int_{|\xi|\geq C(\eta)N(t)} |\xi||\hat{u}(t,\xi)|^2\,\mathrm{d}\xi \leq \eta.$$

We refer to $N(t)$ as the frequency scale function for the solution, and C the compactness modulus function.

Remark 1.2. *By the Arzela-Ascoli theorem, a family of functions is precompact in $\dot{H}_x^{1/2}(\mathbb{R}^5)$ if and only if it is norm-bounded and there exists a compactness modulus function C so that*

$$\int_{|x|\geq C(\eta)} \big||\nabla|^{\frac{1}{2}} f(x)\big|^2 \, \mathrm{d}x + \int_{|\xi|\geq C(\eta)} |\xi||\hat{f}(\xi)|^2 \, \mathrm{d}\xi \leq \eta$$

for all functions in the family and all $\eta > 0$. Thus, u is almost periodic modulo scaling if and only if there exists a compact subset K of $\dot{H}_x^{1/2}(\mathbb{R}^5)$ such that

$$\big\{\, u(t) : t \in I \,\big\} \subseteq \big\{\, \lambda^{-2} f(\lambda^{-1}x) : \lambda \in (0, +\infty), f \in K \,\big\}.$$

By Sobolev's embedding theorem, any solution $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ to (1.1) that is almost periodic modulo scaling also satisfies

$$\int_{|x|\geq C(\eta)/N(t)} |u(t,x)|^{\frac{5}{2}} \, \mathrm{d}x \leq \eta \tag{1.5}$$

for all $t \in I$ and all $\eta > 0$.

By the compactness modulo scaling, there also exists a function $c : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that

$$\int_{|x|\leq c(\eta)/N(t)} \big||\nabla|^{\frac{1}{2}} u(t,x)\big|^2 \, \mathrm{d}x + \int_{|\xi|\leq c(\eta)N(t)} |\xi||\hat{u}(t,\xi)|^2 \, \mathrm{d}\xi \leq \eta \tag{1.6}$$

for all $t \in I$ and all $\eta > 0$.

We now present the process of reduction. If Theorem 1.2 failed, then there must be an almost periodic solution. More precisely, we have:

Theorem 1.3. *Suppose Theorem 1.2 failed for radially symmetric data. Then there exists a maximal life-span solution $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ to (1.1) with $\sup_t \big\||\nabla|^{\frac{1}{2}} u\big\|_2 < \frac{\sqrt{6}}{3}\big\||\nabla|^{\frac{1}{2}} Q\big\|_2$. u is almost periodic modulo scaling, blows up both forward and backward. Moreover, the frequency scale function $N(t)$ and the maximal life-span I match one of the following scenarios :*

I. *(Finite-time blowup) Either $|\inf I| < \infty$ or $\sup I < \infty$.*

II. *(Low-to-high cascade) $I = \mathbb{R}$,*

$$\inf N(t) \geq 1 \quad \text{for all } t \in \mathbb{R}, \quad \text{and} \quad \limsup_{t\to+\infty} N(t) = +\infty.$$

III. *(Soliton-like solution) $I = \mathbb{R}$, $N(t) \equiv 1$ for all $t \in \mathbb{R}$.*

The delicate relationship between the frequency scale function and the maximal life-span for almost periodic solution was first discovered by Killip, Tao, and Visan in [13] for mass-critical NLS. The argument was adapted to the energy-critical case in [14]. This latter argument is directly applicable to the setting of this paper.

To prove Theorem 1.2, it suffices to preclude the three scenarios in Theorem 1.3. We adapt ideas in [13], [14]. However, when precluding the finite-time blowup, Plancherel's

theorem and Hardy's inequality are not enough to obtain a decay for the localized mass, especially for large scales, as we are working in the fractional Sobolev space. To surmount this, we take advantage of the intrinsic description of fractional derivatives, estimate the integral formula in cases according to the spatial scales. Some negative regularity is needed for disproving the rest two scenarios, and our discussions are somewhat involved due to the nonlocal nonlinearity and low regularity. We shall make full use of the frequency localization. For instance, in the proof of Lemma 6.1, we should firstly use Bernstein's inequality to obtain a positive gain in estimating the high frequency components and the medium frequency components, such that the Gronwall's inequality is applicable. What we would also like to emphasize in particular is that as the $\dot{H}^{1/2}$-critical equation enjoys no conservation law, beside proving the negative regularity, we have to gain additional regularity of at least 1 order differentiability, which means that the soliton-like solution has conserved energy; and thus allows us to apply virial-type argument to disprove it. We also obtain the local spacetime bounds in terms of the frequency scale function for all $\dot{H}^{1/2}$-admissible pairs and of those L^2-admissible pairs (q, r) with $q \geq 3$, $r \leq 30/11$.

The following lemma plays an important role in proving the negative and additional regularity. See [28] for a proof.

Lemma 1.1. *Let u be an almost periodic solution to* (1.1) *on its maximal life-span I. Then, for all $t \in I$*

$$\begin{aligned} u(t) &= \lim_{T \nearrow \sup I} i \int_t^T e^{i(t-t')\Delta} F(u(t'))\, \mathrm{d}t' \\ &= -\lim_{T \searrow \inf I} \int_T^t e^{i(t-t')\Delta} F(u(t'))\, \mathrm{d}t' \end{aligned} \tag{1.7}$$

as weak limits in $\dot{H}^{1/2}_x$.

The rest of paper is organized as follows. In Section 2, we list out some notations and known results that we use repeatedly in the paper. In Section 3, the sharp constant for a Hardy-Littlewood-Sobolev type inequality is obtained, and a sufficient condition for global existence of (1.1) with finite energy initial data is given. In Section 4, we first prove a Palais-Smale condition modulo scaling, and then Theorem 1.3. In Section 5, we preclude the finite-time blowup scenario. In Section 6, we prove the negative regularity for global case. In Section 7, we disprove the low-to-high cascade. In Section 8, we prove an additional regularity for the soliton-like solution. In Section 9, we preclude the soliton-like solution. In Section 10, we prove Proposition 1.1.

2 Preliminaries

2.1 Notations

For any spacetime slab $I \times \mathbb{R}^5$, we use $L_t^q L_x^r(I \times \mathbb{R}^d)$ to denote the Banach space with norm

$$\|u\|_{L_t^q L_x^r} := \left(\int_I \left(\int_{\mathbb{R}^d} |u(t,x)|^r \,\mathrm{d}x \right)^{q/r} \mathrm{d}t \right)^{1/q},$$

with the usual modifications when q or r are infinity. When $q = r$ we abbreviate $L_t^q L_x^r$ as $L_{t,x}^q$.

We use the 'Japanese bracket' convention $\langle x \rangle := (1 + |x|^2)^{1/2}$.

We use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some constant $C > 0$. If C depends on some parameters, we will indicate this with subscripts; for example, $X \lesssim_u Y$ denote the assertion that $X \leq C_u Y$ for some C_u depending on u. We denote by $X\pm$ any quantity of the form $X \pm \varepsilon$ for any $\varepsilon > 0$. we define the Fourier transform on $\mathbb{R}^d$ by

$$\hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \,\mathrm{d}x.$$

For $s \in \mathbb{R}$, we define the fractional differential/integral operators

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi)$$

and the homogeneous Sobolev norm

$$\|f\|_{\dot{H}_x^s(\mathbb{R}^d)} := \left\| |\nabla|^s f \right\|_{L_x^2(\mathbb{R}^d)}.$$

The next following lemma is a form of Gronwall's inequality that we will use to handle some bootstrap argument below.

Lemma 2.1 (Gronwall's inequality)**.** *Given $\gamma > 0$, $0 < \eta < \frac{1}{2}(1 - 2^{-\gamma})$ and $\{b_k\} \in l^\infty(\mathbb{Z}^+)$. Let $\{x_k\} \in l^\infty(\mathbb{Z}^+)$ be a non-negative sequence obeying*

$$x_k \leq b_k + \eta \sum_{l=0}^{\infty} 2^{-\gamma|k-l|} x_l \quad \text{for all } k \geq 0.$$

Then

$$x_k \lesssim \sum_{l=0}^{\infty} r^{|k-l|} b_l \quad \text{for all } k \geq 0 \tag{2.1}$$

for some $r = r(\eta) \in (2^{-\gamma}, 1)$. Moreover, $r \downarrow 2^{-\gamma}$ as $\eta \downarrow 0$.

2.2 Basic harmonic analysis

Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\, \xi \in \mathbb{R}^d : |\xi| \leq \frac{11}{10} \,\}$ and equal to 1 on the ball $\{\, \xi \in \mathbb{R}^d : |\xi| \leq 1 \,\}$. For each number $N > 0$, we define the Fourier

multipliers

$$\begin{aligned}
\widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi), \\
\widehat{P_{> N} f}(\xi) &:= (1-\varphi(\xi/N))\hat{f}(\xi), \\
\widehat{P_N f}(\xi) &:= \psi(\xi/N)\hat{f}(\xi) = (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi)
\end{aligned}$$

and similarly $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

for $M < N$. We will use these multipliers when M and N are dyadic numbers; in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be the power of 2. Note that, P_N is not truly a projection; to get around this, define

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}.$$

These obey $\tilde{P}_N P_N = P_N \tilde{P}_N = P_N$.

The Littlewood-Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many many times. First, we introduce

Lemma 2.2 (Bernstein)**.** *For $1 \leq p \leq q \leq \infty$,*

$$\begin{aligned}
\big\| |\nabla|^{\pm s} P_N f \big\|_{L_x^q(\mathbb{R}^d)} &\sim N^{\pm s} \|P_N f\|_{L_x^p(\mathbb{R}^d)}, \\
\|P_{\leq N} f\|_{L_x^q(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_{\leq N} f\|_{L_x^p(\mathbb{R}^d)}, \\
\|P_N f\|_{L_x^q(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L_x^p(\mathbb{R}^d)}.
\end{aligned}$$

We also need the following fractional Leibniz rule, [11].

Lemma 2.3 (Fractional Leibniz rule)**.** *Let $\alpha \in (0,\,1)$, $\alpha_1,\,\alpha_2 \in [0, \alpha]$ with $\alpha = \alpha_1 + \alpha_2$. Let $1 < p,\, p_1,\, p_2,\, q,\, q_1,\, q_2 < \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\big\| D^\alpha(fg) - gD^\alpha f - fD^\alpha g \big\|_{L_t^q L_x^p} \lesssim \big\| D^{\alpha_1} f \big\|_{L_t^{q_1} L_x^{p_1}} \|D^{\alpha_2} g\|_{L_t^{q_2} L_x^{p_2}}.$$

If $\alpha_1 = 0$, $q_1 = \infty$ is allowed.

2.3 Strichartz's estimates

Let $e^{it\Delta}$ be the free Schrödinger evolution. From the explicit formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y)\,\mathrm{d}y,$$

we deduce the standard dispersive inequality

$$\|e^{it\Delta} f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1(\mathbb{R}^d)}$$

for all $t \neq 0$.

Finer bounds on (frequency localized) linear propagator can be derived using stationary phase:

Lemma 2.4 (Kernel estimates, [13]). *For any $m \geq 0$, the kernel of the linear propagator obeys the following estimates:*

$$|(P_N e^{it\Delta})(x,y)| \lesssim_m \begin{cases} |t|^{-d/2}, & |x-y| \sim N|t| \\ \dfrac{N^d}{|Nt|^m \langle N|x-y|\rangle^m}, & \textit{otherwise} \end{cases}$$

for $|t| \geq N^{-2}$ and

$$|(P_N e^{it\Delta})(x,y)| \lesssim_m N^d \langle N|x-y|\rangle^{-m}$$

for $|t| \leq N^{-2}$.

The standard Strichartz's estimate reads:

Lemma 2.5 (Strichartz). *Let $k \geq 0$, $d \geq 3$. Let I be a compact time interval, $t_0 \in I$. Then the function u defined by*

$$u(t) := e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^{t} e^{i(t-t')\Delta} f(t')\, \mathrm{d}t' \tag{2.2}$$

obeys

$$\|u\|_{\dot{S}^k(I)} \lesssim \|u(t_0)\|_{\dot{H}^k_x} + \|f\|_{\dot{N}^k(I)}$$

for any $t_0 \in I$, where $\dot{S}^k(I)$ is the Strichartz norm, and $\dot{N}^k(I)$ is its dual norm.

Proof. See, for example, [6], [8]. For a textbook treatment, see [20].

We also need the following weighted Strichartz's inequality. It is very useful in regions of space far from the origin.

Lemma 2.6 (Weighted Strichartz, [15]). *Let I be an interval, $t_0 \in I$, $u_0 \in L^2_x(\mathbb{R}^d)$, $f \in L^{2(d+2)/(d+4)}_{t,x}(I \times \mathbb{R}^d)$ be radially symmetric. Then the function u defined by (2.2) obeys the estimate*

$$\left\| |x|^{\frac{2(d-1)}{q}} u \right\|_{L^q_t L^{\frac{2q}{q-4}}_x (I\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^d)} + \|f\|_{L^2_t L^{2d/(d+2)}_x (I\times\mathbb{R}^d)}$$

for all $4 \leq q \leq \infty$.

2.4 In/out decomposition

For a radially symmetric function f, we define the projection onto outgoing spherical waves by

$$[P^+ f](r) = \frac{1}{2}\int_0^\infty r^{\frac{2-d}{2}} H^{(1)}_{\frac{d-2}{2}}(kr) \hat{f}(k) k^{\frac{d}{2}}\, \mathrm{d}k$$

and the projection onto incoming spherical waves by

$$[P^- f](r) = \frac{1}{2}\int_0^\infty r^{\frac{2-d}{2}} H^{(2)}_{\frac{d-2}{2}}(kr)\hat{f}(k)k^{\frac{d}{2}}\,\mathrm{d}k$$

where $H^{(1)}_{\frac{d-2}{2}}$ denotes the Hankle function of the first kind with order $\frac{d-2}{2}$ and $H^{(2)}_{\frac{d-2}{2}}$ denotes the Hankle function of the second kind with the same order. We write $P_N^{\pm}$ for the product $P^{\pm}P_N$, then we have

Lemma 2.7 (Kernel estimates, [15]). *For $|x| \gtrsim N^{-1}$ and $|t| \gtrsim N^{-2}$, the integral kernel obeys*

$$\left|[P_N^{\pm} e^{\mp it\Delta}](x,y)\right| \lesssim \begin{cases} (|x||y|)^{-\frac{d-1}{2}}|t|^{-\frac{1}{2}}, & |y|-|x| \sim N|t| \\ \dfrac{N^d}{(N|x|)^{\frac{d-1}{2}}\langle N|y|\rangle^{\frac{d-1}{2}}}\langle N^2t + N|x| - N|y|\rangle^{-m}, & \textit{otherwise} \end{cases}$$

for any $m \geq 0$. For $|x| \gtrsim N^{-1}$ and $|t| \lesssim N^{-2}$, the integral kernel obeys

$$\left|[P_N^{\pm} e^{\mp it\Delta}](x,y)\right| \lesssim \frac{N^d}{(N|x|)^{\frac{d-1}{2}}\langle N|y|\rangle^{\frac{d-1}{2}}}\langle N|x| - N|y|\rangle^{-m}$$

for any $m \geq 0$.

Lemma 2.8 (Properties of $P^{\pm}$, [15]). *We have:*

- *$P^+ + P^-$ acts as the identity on $L^2_{rad}(\mathbb{R}^d)$.*
- *Fix $N > 0$, for any radially symmetric function $f \in L^2_x(\mathbb{R}^d)$,*

$$\|P^{\pm}P_{\geq N} f\|_{L^2_x(|x| \geq \frac{1}{100N})} \lesssim \|f\|_{L^2_x(\mathbb{R}^d)},$$

with an N-independent constant.

2.5 Concentration compactness

In this subsection we record the linear profile decomposition statement due to Shao [26]. We first recall the symmetries of the solutions to equation (1.1) which fix the initial surface $t = 0$.

Definition 2.1 (Symmetry group). *For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^5$, and scaling parameter $\lambda > 0$, we define the unitary transformation $g_{\theta,x_0,\lambda} : \dot{H}^{1/2}_x(\mathbb{R}^5) \mapsto \dot{H}^{1/2}_x(\mathbb{R}^5)$ by*

$$[g_{\theta,x_0,\lambda} f](x) := \lambda^{-2} e^{i\theta} f(\lambda^{-1}(x - x_0)).$$

Let G denotes the collection of such transformations. For a function $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$, define $T_{g_{\theta,x_0,\lambda}} u : \lambda^2 I \times \mathbb{R}^5 \mapsto \mathbb{C}$ by

$$[T_{g_{\theta,x_0,\lambda}} u](t,x) := \lambda^{-2} e^{i\theta} u(\lambda^{-2}t, \lambda^{-1}(x - x_0))$$

where $\lambda^2 I := \{\, \lambda^2 t : \, t \in I \,\}$.

Let $G_{rad} \subset G$ *denotes the collection of transformations in* G *which preserves radial symmetry, or more precisely*

$$G_{rad} := \{\, g_{\theta,0,\lambda} : \theta \in \mathbb{R}/2\pi\mathbb{Z},\ \lambda > 0 \,\}.$$

Remark 2.1. *u is a maximal life-span solution to* (1.1) *if and only if $T_g u$ is a maximal life-span solution to* (1.1)*. Moreover,*

$$\|T_g u\|_{\dot{H}^{1/2}_x(\mathbb{R}^5)} = \|u\|_{\dot{H}^{1/2}_x(\mathbb{R}^5)}, \quad \|T_g u\|_{S(\lambda^2 I)} = \|u\|_{S(I)}, \quad \text{for all } g \in G.$$

We are now ready to state the linear profile decomposition.

Lemma 2.9 (Linear profiles, [26])**.** *Let $\{u_n\}_{n\geq 1}$ be a bounded sequence of functions in $\dot{H}^{1/2}_x(\mathbb{R}^5)$. Then after passing to a subsequence if necessary, there exist a sequence of functions $\{\phi^j\}_{j\geq 1} \subset \dot{H}^{1/2}_x(\mathbb{R}^5)$, group elements $g_n^j \in G$, and times $t_n^j \in \mathbb{R}$ such that we have the decomposition*

$$u_n = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta} \phi^j + \omega_n^J \tag{2.3}$$

for all $J \geq 1$; $\omega_n^J \in \dot{H}^{1/2}_x(\mathbb{R}^5)$ obeying

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e^{it\Delta}\omega_n^J\|_{L_t^3 L_x^{15/4}(\mathbb{R}\times\mathbb{R}^5)} = 0. \tag{2.4}$$

Moreover, for any $j' \neq j$, we have the following orthogonal property

$$\lim_{n\to\infty} \left(\frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{|x_n^j - x_n^{j'}|}{\lambda_n^j} + \frac{|t_n^j - t_n^{j'}|}{(\lambda_n^j)^2} \right) = 0. \tag{2.5}$$

For any $J \geq 1$

$$\lim_{n\to\infty} \left[\big\||\nabla|^{\frac{1}{2}} u_n\big\|_2^2 - \sum_{j=1}^{J} \big\||\nabla|^{\frac{1}{2}} \phi^j\big\|_2^2 - \big\||\nabla|^{\frac{1}{2}} \omega_n^J\big\|_2^2 \right] = 0. \tag{2.6}$$

When $\{u_n\}$ is assumed to be radially symmetric, one can choose ϕ^j, ω_n^J to be radially symmetric and $g_n^j \in G_{rad}$.

The error term also satisfies the following lemma

Lemma 2.10. *For all $J \geq 1$, $1 \leq j \leq J$, the sequence $e^{-it_n^j\Delta}[(g_n^j)^{-1}\omega_n^J]$ converges weakly to zero in $\dot{H}^{1/2}_x(\mathbb{R}^5)$ as $n \to \infty$.*

Proof. The proof is an analogue to that in [14], [9].

We end this section with a perturbation theorem

Theorem 2.1 (Long time perturbation theory)**.** *Let $I \subset \mathbb{R}$ be a compact time interval and let $t_0 \in I$. Let $\tilde{u} : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ be a near-solution to* (1.1) *in the sense that*

$$i\partial_t \tilde{u} + \Delta \tilde{u} = F(\tilde{u}) + e$$

for some function e. Suppose $\tilde{u}$ satisfies

$$\sup_{t\in I} \|\tilde{u}\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} \leq A, \quad \|\tilde{u}\|_{S(I)} \leq M, \quad \|\tilde{u}\|_{X(I)} < +\infty,$$

for some constant M, $A > 0$. Assume also that

$$\begin{aligned} \|u_0 - \tilde{u}(t_0)\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} &\leq A', \\ \big\||\nabla|^{1/2} e\big\|_{L_t^1 L_x^2(I\times\mathbb{R}^5)} &\leq \varepsilon, \\ \big\|e^{i(t-t_0)\Delta}(u_0 - \tilde{u}(t_0))\big\|_{S(I)} &\leq \varepsilon. \end{aligned}$$

Then, there exists a solution $u : I \times \mathbb{R}^5$ to (1.1) *with $u(t_0) = u_0$ such that*

$$\sup_{t\in I} \|u - \tilde{u}(t)\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} + \|u - \tilde{u}\|_{S(I)} + \|u - \tilde{u}\|_{X(I)} \leq \varepsilon.$$

3 Sharp constant for a Hardy-Littlewood-Sobolev type inequality

In this section we find the best constant to the following Hardy-Littlewood-Sobolev type inequality

$$\iint_{\mathbb{R}^5\times\mathbb{R}^5} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^3} \,\mathrm{d}x \,\mathrm{d}y \leq C_5 \big\||\nabla|^{\frac{1}{2}} u\big\|_2^2 \|\nabla u\|_2^2, \tag{3.1}$$

and obtain a sufficient condition for global existence of equation (1.1) with initial data in $\dot{H}_x^1(\mathbb{R}^5) \cap \dot{H}_x^{1/2}(\mathbb{R}^5)$. We find that the best constant $C_5 = 2\big\||\nabla|^{\frac{1}{2}} Q\big\|_2^{-2}$, where Q is the solution to (1.4). The approach is essentially from [30].

Consider the Weinstein functional

$$J(u) = \frac{\big\||\nabla|^{\frac{1}{2}} u\big\|_2^2 \|\nabla u\|_2^2}{\int_{\mathbb{R}^5} (|\cdot|^{-3} * |u|^2)|u|^2 \,\mathrm{d}x}, \qquad \forall u \in \dot{H}_x^1(\mathbb{R}^5) \cap \dot{H}_x^{1/2}(\mathbb{R}^5).$$

First observe that if we set $u_{a,b} = au(bx)$, then

$$J(u_{a,b}) = J(u), \qquad \big\||\nabla|^{\frac{1}{2}} u_{a,b}\big\|_2^2 = a^2 b^{-4} \big\||\nabla|^{\frac{1}{2}} u\big\|_2^2, \qquad \|\nabla u_{a,b}\|_2^2 = a^2 b^{-3} \|\nabla u\|_2^2.$$

Theorem 3.1.

$$C_5^{-1} = \inf_{u\in \dot{H}_x^1(\mathbb{R}^5)\cap \dot{H}_x^{1/2}(\mathbb{R}^5)\setminus\{0\}} J(u)$$

can be obtained at some $Q \in \dot{H}_x^1(\mathbb{R}^5) \cap \dot{H}_x^{1/2}(\mathbb{R}^5)$. In addition, $C_5 = 2\big\||\nabla|^{\frac{1}{2}} Q\|_2^{-2}$.

Before proving the theorem, we present some compactness tools.

Lemma 3.1 (Radial Lemma). *Let $d \geq 3$, $u \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d) \cap \dot{H}^{1/2}_{\mathrm{rad}}(\mathbb{R}^d)$ be a radially symmetric function. Then*

$$\sup_{x \in \mathbb{R}^d} |x|^{\frac{2d-3}{4}} |u(x)| \lesssim \big\| |\nabla|^{\frac{1}{2}} u \big\|_2^{\frac{1}{2}} \|\nabla u\|_2^{\frac{1}{2}}. \tag{3.2}$$

Proof. Suppose first $u \in C_c^\infty(\mathbb{R}^d)$. We have

$$\begin{aligned} r^{\frac{2d-3}{2}} u(r)^2 &= - \int_r^\infty \frac{\mathrm{d}}{\mathrm{d}s} \big(s^{\frac{2d-3}{2}} u(s)^2 \big) \mathrm{d}s \\ &\leq -2 \int_r^\infty s^{\frac{2d-3}{2}} u(s) u'(s) \mathrm{d}s \\ &\lesssim \big\| |x|^{-\frac{1}{2}} u \|_2 \big\| \nabla u \|_2, \end{aligned}$$

(3.2) follows from Hardy's inequality. The general case then follows by the density argument.

Lemma 3.2 (Compactness Lemma).

$$\dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d) \cap \dot{H}^{1/2}_{\mathrm{rad}}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \quad \textit{for all} \quad \frac{2d}{d-1} < p < \frac{2d}{d-2}.$$

Proof. Let $\{u_k\}$ be a bounded sequence in $\dot{H}^1_{\mathrm{rad}} \cap \dot{H}^{1/2}_{\mathrm{rad}}$, then by the weak compactness principle, there exists $u \in \dot{H}^1_{\mathrm{rad}} \cap \dot{H}^{1/2}_{\mathrm{rad}}$ such that $u_k \rightharpoonup u$ weakly in $\dot{H}^1_{\mathrm{rad}} \cap \dot{H}^{1/2}_{\mathrm{rad}}$.

For $\varepsilon > 0$, let $R > 0$ to be chosen later. Given p as in the statement, we have

$$\begin{aligned} \|u_k - u\|_{L^p(\mathbb{R}^d)} &\leq \|u_k - u\|_{L^p(B_R)} + \|u_k - u\|_{L^p(\{\, x : |x| > R \,\})} \\ &\leq \|u_k - u\|_{L^p(B_R)} + \|u_k - u\|_{L^\infty(\{\, x : |x| > R \,\})}^{\frac{p(d-1)-2d}{(d-1)p}} \|u_k - u\|_{L^{\frac{2d}{d-1}}(\mathbb{R}^d)}^{\frac{2d}{(d-1)p}}. \end{aligned}$$

By Lemma 3.1, we first choose R large enough so that

$$\|u_k - u\|_{L^\infty(\{\, x : |x| > R \,\})}^{\frac{p(d-1)-2d}{(d-1)p}} \|u_k - u\|_{L^{\frac{2d}{d-1}}(\mathbb{R}^d)}^{\frac{2d}{(d-1)p}} \leq \frac{\varepsilon}{2}.$$

On the other hand, it follows from Rellich's compactness lemma that

$$\|u_k - u\|_{L^p(B_R)} \leq \frac{\varepsilon}{2}$$

for large k and so $\|u_k - u\|_{L^p(\mathbb{R}^d)} \leq \varepsilon$. This proves the lemma.

Proof of Theorem 3.1 . Since $J(u) \geq 0$, we may find a minimizing sequence $\{u_k\} \subset \dot{H}^1 \cap \dot{H}^{1/2}$ such that

$$C_5^{-1} = \inf J(u) = \lim_{k \to \infty} J(u_k).$$

By symmetric rearrangement technique, we may assume $u_k > 0$ and is radially symmetric for all k.

Set $a_k = \big\||\nabla|^{\frac12} u_k\big\|_2^3/\|\nabla u_k\|_2^4$, $b_k = \big\||\nabla|^{\frac12} u_k\big\|_2^2/\|\nabla u_k\|_2^2$, and $Q_k = a_k u(b_k x)$. Then $Q_k \geq 0$, is radially symmetric. Moreover, we have

$$\big\||\nabla|^{\frac12} Q_k\big\|_2 = \|\nabla Q_k\|_2 = 1, \quad \lim_{k\to\infty} J(Q_k) = C_5^{-1}.$$

Since $\{Q_k\} \subset \dot H^1_{\rm rad} \cap \dot H^{1/2}_{\rm rad}$ is uniformly bounded, up to a subsequence, $Q_k \rightharpoonup Q^*$ in $\dot H^1_{\rm rad} \cap \dot H^{1/2}_{\rm rad}$, and $\big\||\nabla|^{\frac12} Q^*\big\|_2 \leq 1$, $\|\nabla Q^*\|_2 \leq 1$. From Lemma 3.2, $Q_k \to Q^*$ in $L^p(\mathbb{R}^5)$ for $\frac52 < p < \frac{10}{3}$. Furthermore, we have

$$\iint_{\mathbb{R}^5\times\mathbb{R}^5} \frac{|Q_k(x)|^2|Q_k(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y \longrightarrow \iint_{\mathbb{R}^5\times\mathbb{R}^5} \frac{|Q^*(x)|^2|Q^*(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y \quad \text{as}\, k\to\infty.$$

This is easily checked by a direct computation using the Hardy-Littlewood-Sobolev inequality.

Thus

$$C_5^{-1} \leq J(Q^*) \leq \frac{1}{\int_{\mathbb{R}^5}(|\cdot|^{-3} * |Q^*|^2)|Q^*|^2\,\mathrm{d}x} = \lim_{k\to\infty} J(Q_k) = C_5^{-1}.$$

This implies that $\big\||\nabla|^{\frac12} Q^*\big\|_2^2\|\nabla Q^*\|_2^2 = 1$, which further gives $\big\||\nabla|^{\frac12} Q^*\big\|_2 = \|\nabla Q^*\|_2 = 1$.

Since Q^* is a minimizer, it satisfies the Euler-Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} J(Q^* + \varepsilon\phi) = 0 \quad \text{for all}\, \phi \in C_0^\infty(\mathbb{R}^5).$$

Taking into account the fact that $\big\||\nabla|^{\frac12} Q^*\big\|_2 = \|\nabla Q^*\|_2 = 1$, we have

$$-\Delta Q^* + (-\Delta)^{1/2} Q^* - 2C_5^{-1}(|\cdot|^{-3} * |Q^*|^2)Q^* = 0.$$

Let $Q^* = \sqrt{C_5/2}Q$, then Q solves (1.4).

By the fact that $\big\||\nabla|^{\frac12} Q^*\big\|_2 = 1$, it yields $C_5 = 2\big\||\nabla|^{\frac12} Q\big\|_2^{-2}$. $\square$

Proposition 3.1. *Let* $u_0 \in \dot H^1_x(\mathbb{R}^5) \cap \dot H^{1/2}_x(\mathbb{R}^5)$. *Suppose* $\sup_t \||\nabla|^{\frac12} u\|_2 < \||\nabla|^{\frac12} Q\|_2$, *then the solution to* (1.1) *is global.*

Proof. It is a consequence of the energy conservation

$$E(u(t)) = \frac12\int_{\mathbb{R}^5} |\nabla u|^2\,\mathrm{d}x - \frac14\iint_{\mathbb{R}^5\times\mathbb{R}^5} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y,$$

and (3.1).

4 Reduction to almost periodic solution

In this section we will prove Theorem 1.3. The main step toward this end is to prove a Palais-Smale condition modulo scaling.

For any $A > 0$, define

$$L(A) = \sup\left\{ \|u\|_{S(I)} : \ u : I \times \mathbb{R}^5 \mapsto \mathbb{C} \, \text{such that } \sup_{t \in I} \|u\|_{\dot{H}_x^{1/2}} \leq A \right\}.$$

Here, the supremum is taken over all solutions $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ to (1.1) satisfying $\sup_{t\in I} \|u\|_{\dot{H}_x^{1/2}} \leq A$. Note that $L(A)$ is non-decreasing and left-continuous. On the other hand, from Theorem 1.1,

$$L(A) \lesssim A \quad \text{for} \quad A \leq \delta_0,$$

where δ_0 is the threshold from the small data global well-posedness theory. Theorem 1.2 states that for each $A < \frac{\sqrt{6}}{3}\|Q\|_{\dot{H}^{1/2}}$, $L(A) < \infty$. Therefore, if Theorem 1.2 failed, there exists $\delta_0 < A_c < \frac{\sqrt{6}}{3}\|Q\|_{\dot{H}^{1/2}}$ such that $L(A) < +\infty$ for $A < A_c$, $L(A) = +\infty$ for $A \geq A_c$.

Convention: In this section and the rest sections, we write $|x|^{-3} *$ as $|\nabla|^{-2}$ since they are equivalent up to a constant. Moreover, we ignore the distinction between a function and its conjugation as they make no difference in our discussion.

4.1 Palais-Smale condition modulo scaling

Proposition 4.1. *Let $u_n : I_n \times \mathbb{R}^5 \mapsto \mathbb{C}$ be a sequence of solutions to* (1.1) *such that*

$$\limsup_{n\to\infty} \sup_{t \in I_n} \|u_n(t)\|_{\dot{H}_x^{1/2}} = A_c. \tag{4.1}$$

Let $t_n \in I_n$ be a time sequence such that

$$\lim_{n\to\infty} \|u_n\|_{S(-\infty,\, t_n)} = \lim_{n\to\infty} \|u_n\|_{S(t_n,\, \infty)} = \infty.$$

Then there exists a subsequence of $u_n(t_n)$, which converges in $\dot{H}_x^{1/2}(\mathbb{R}^5)$ modulo scaling.

The proof of this Proposition is achieved through several steps.

Proof. By time-translation invariant of (1.1), we may set $t_n = 0$ for all $n \geq 1$. Then

$$\lim_{n\to\infty} \|u_n\|_{S(-\infty,\, 0)} = \lim_{n\to\infty} \|u_n\|_{S(0,\, \infty)} = \infty. \tag{4.2}$$

Now applying Lemma 2.9 to the sequence $u_n(0)$, and up to a subsequence, we obtain a decomposition

$$u_n(0) = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta} \phi^j + \omega_n^J$$

for any $J \geq 1$, $n \geq 1$.

By passing to a further subsequence, we may assume t_n^j converges to some $t^j \in [-\infty, +\infty]$ for each j. If t^j is finite, then replacing ϕ^j by $e^{it^j\Delta}\phi^j$, we may set $t^j = 0$. Adding $e^{it_n^j\Delta}\phi^j - \phi^j$ to the error term ω_n^J, we may assume $t_n^j \equiv 0$. Thus, we only need to deal with $t_n^j \equiv 0$ and $t_n^j \to \pm\infty$.

For each ϕ^j and t_n^j, define nonlinear profile $v^j : I^j \times \mathbb{R}^5 \mapsto \mathbb{C}$ as follows:

- If $t_n^j \equiv 0$, then v^j is the maximal life-span solution to (1.1) with initial data $v^j(0) = \phi^j$.
- If $t_n^j \to \infty$, then v^j is the maximal life-span solution to (1.1) that scatters forward to $e^{it\Delta}\phi^j$.
- If $t_n^j \to -\infty$, then v^j is the maximal life-span solution to (1.1) that scatters backward to $e^{it\Delta}\phi^j$.

For each j, $n \geq 1$, define $v_n^j : I_n^j \times \mathbb{R}^5 \mapsto \mathbb{C}$ by

$$v_n^j(t) := T_{g_n^j}[v^j(\cdot + t_n^j)](t),$$

where $I_n^j := \{\, t \in \mathbb{R} : (\lambda_n^j)^{-2}t + t_n^j \in I^j \,\}$. Then for each j, v_n^j is also a maximal life-span solution to (1.1) with initial data $v_n^j(0) = g_n^j v^j(t_n^j)$, and with maximal life-span $I_n^j = (-T_{n,j}^-,\, T_{n,j}^+)$, $-\infty \leq -T_{n,j}^- < 0 < T_{n,j}^+ \leq +\infty$.

With these preliminaries out of the way, we first have

Step 1: There exists $J_0 \geq 1$ such that, for all $j \geq J_0$, n sufficiently large

$$\sup_{t\in\mathbb{R}} \|v_n^j(t)\|_{\dot{H}_x^{1/2}} + \|v_n^j\|_{S(\mathbb{R})} + \|v_n^j\|_{X(\mathbb{R})} \lesssim \|\phi^j\|_{\dot{H}_x^{1/2}}. \tag{4.3}$$

Proof. From (2.6), there exists $J_0 \geq 1$ such that for sufficiently large n

$$\|\phi^j\|_{\dot{H}_x^{1/2}} \leq \delta_0 \quad \text{for all} \quad j \geq J_0$$

where δ_0 is the threshold from the small data theory. Hence, by Theorem 1.1, v_n^j is global and

$$\sup_{t\in\mathbb{R}} \|v_n^j\|_{\dot{H}_x^{1/2}} + \|v_n^j\|_{X(\mathbb{R})} + \|v_n^j\|_{S(\mathbb{R})} \lesssim \|\phi^j\|_{\dot{H}_x^{1/2}}.$$

for all $j \geq J_0$ and all n sufficiently large.

Step 2: There exists $1 \leq j_0 < J_0$ such that

$$\limsup_{n\to\infty} \|v_n^{j_0}\|_{S(0,\, T_{n,j_0}^+)} = \infty.$$

Proof. Suppose to the contrary that for all $1 \leq j < J_0$

$$\limsup_{n\to\infty} \|v_n^j\|_{S(0,\, T_{n,j}^+)} \leq M < \infty \tag{4.4}$$

for some $M > 0$. This implies that $T_{n,j}^+ = \infty$ for all $1 \leq j < J_0$ and all sufficiently large n. Given $\eta > 0$, divide $(0, \infty)$ into subintervals I_k such that on each I_k, $\|v_n^j\|_{S(I_k)} \leq \eta$. By Strichartz's estimate, we have for all $1 \leq j < J_0$ and all large n that

$$\|v_n^j\|_{X(0,\infty)} < \infty. \tag{4.5}$$

Indeed, let $\eta > 0$, divide $(0, \infty)$ into subintervals $I_k = [t_k, t_{k+1}]$ such that on each I_k we have $\|v_n^j\|_{S(I_k)} \leq \eta$. Note that, there are at most $\eta^{-1} \times M$ such intervals. Applying the Strichartz estimate

$$\begin{aligned}\|v_n^j\|_{X(I_k)} &\lesssim \|v_n^j(t_k)\|_{\dot{H}_x^{1/2}} + \big\||\nabla|^{\frac{1}{2}} F(v_n^j)\big\|_{L_t^1 L_x^2} \\ &\lesssim A_c + \|v_n^j\|_{S(I_k)}^2 \|v_n^j\|_{X(I_k)}.\end{aligned}$$

If we choose $\eta > 0$ sufficiently small, then

$$\|v_n^j\|_{X(I_k)} \lesssim A_c.$$

Summing over all I_k, we achieve (4.5).

Combining (4.4) with Step 1, and then using (2.6) and (4.1), we have that for all sufficiently large n,

$$\sum_{j \geq 1} \sup_{t \in (0,\infty)} \|v_n^j\|_{\dot{H}_x^{1/2}} + \|v_n^j\|_{S(0,\infty)} + \|v_n^j\|_{X(0,\infty)} \lesssim 1 + A_c. \tag{4.6}$$

Next, we will use perturbation theorem to obtain a bound on $\|u_n\|_{S(0,\,\infty)}$ for n sufficiently large.

Define an approximation to u_n by

$$u_n^J(t) := \sum_{j=1}^{J} v_n^j(t) + e^{it\Delta} \omega_n^J. \tag{4.7}$$

Then, by the definition of nonlinear profile

$$\begin{aligned}\limsup_{n\to\infty} \|u_n^J(0) - u_n(0)\|_{\dot{H}_x^{1/2}} &= \limsup_{n\to\infty} \Big\| \sum_{j=1}^{J} g_n^j v^j(t_n^j) - g_n^j e^{it_n^j \Delta} \phi^j \Big\|_{\dot{H}_x^{1/2}} \\ &\lesssim \limsup_{n\to\infty} \sum_{j=1}^{J} \|v^j(t_n^j) - e^{it_n^j \Delta} \phi^j\|_{\dot{H}_x^{1/2}} = 0.\end{aligned}$$

Note that (2.5) with a few computations yields that for all $j \geq 1$

$$\limsup_{n\to\infty} \|v_n^{j'} v_n^j\|_{S(0,\,\infty)} = 0 \tag{4.8}$$

for any $j' \neq j$.(Such an asymptotic orthogonal property was well developed in [12], [26], we refer to them for details.)

Thus, by (2.4), (4.6) and (4.8)

$$\begin{aligned}\lim_{J\to\infty} \limsup_{n\to\infty} \|u_n^J\|_{S(0,\,\infty)} &\lesssim \lim_{J\to\infty} \limsup_{n\to\infty} \Big(\Big\| \sum_{j=1}^{J} v_n^j \Big\|_{S(0,\,\infty)} + \big\|e^{it\Delta} \omega_n^J\big\|_{S(0,\,\infty)} \Big) \\ &\lesssim \lim_{J\to\infty} \limsup_{n\to\infty} \sum_{j=1}^{J} \|v_n^j\|_{S(0,\,\infty)} \lesssim 1 + A_c. \end{aligned} \tag{4.9}$$

By the same argument as that to derive (4.5) from (4.4), we obtain

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|u_n^J\|_{X(0,\,\infty)}<\infty.$$

Now, we have to verify that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\Big\||\nabla|^{\frac12}\big[(i\partial_t+\Delta)u_n^J+F(u_n^J)\big]\Big\|_{L_t^1L_x^2((0,\infty)\times\mathbb{R}^5)}=0.$$

Using the triangle inequality, we need to show on $(0,\infty)\times\mathbb{R}^5$ that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\Big\||\nabla|^{\frac12}\Big[\sum_{j=1}^J F(v_n^j)-F(\sum_{j=1}^J v_n^j)\Big]\Big\|_{L_t^1L_x^2}=0 \tag{4.10}$$

and

$$\lim_{J\to\infty}\limsup_{n\to\infty}\Big\||\nabla|^{\frac12}\big(F(u_n^J-e^{it\Delta}\omega_n^J)-F(u_n^J)\big)\Big\|_{L_t^1L_x^2}=0. \tag{4.11}$$

We first consider (4.10). By expanding out the nonlinearity

$$\begin{aligned}&\Big||\nabla|^{\frac12}\Big[\sum_{j=1}^J F(v_n^j)-F(\sum_{j=1}^J v_n^j)\Big]\Big|\\ &\le\sum_{j_1,j_2,j_3=1}^J\Big||\nabla|^{\frac12}\big[\big(|\nabla|^{-2}(v_n^{j_1}v_n^{j_2})\big)v_n^{j_3}\big]\Big|,\end{aligned}$$

where at least two of j_1,j_2,j_3 are different.

Note that the nonlocal action (i.e. convolution) break up the spatial orthogonality, whereas time orthogonality will be preserved. Recalling the radial assumption, we may assume $j_2\neq j_1$. Thus, using the fractional Leibniz rule, Hölder's inequality,the Hardy-Littlewood-Sobolev inequality, and (4.8), we obtain on $(0,\infty)\times\mathbb{R}^5$ that

$$\begin{aligned}&\lim_{J\to\infty}\limsup_{n\to\infty}\Big\||\nabla|^{\frac12}\Big[\sum_{j=1}^J F(v_n^j)-F(\sum_{j=1}^J v_n^j)\Big]\Big\|_{L_t^1L_x^2}\\
&\lesssim_J\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{j_1,j_2,j_3=1}^J\Big(\big\||\nabla|^{\frac12}\big(|\nabla|^{-2}(v_n^{j_1}v_n^{j_2})\big)v_n^{j_3}\big\|_{L_t^1L_x^2}+\big\|\big(|\nabla|^{-2}(v_n^{j_1}v_n^{j_2})\big)|\nabla|^{\frac12}v_n^{j_3}\big\|_{L_t^1L_x^2}\Big)\\
&\lesssim_J\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{j_1,j_2,j_3=1}^J\Big(\big\||\nabla|^{\frac12}\big(|\nabla|^{-2}(v_n^{j_1}v_n^{j_2})\big)\big\|_{L_t^{\frac32}L_x^{\frac{30}{7}}}\|v_n^{j_3}\|_{S(0,\infty)}\\
&\qquad\qquad+\big\||\nabla|^{-2}(v_n^{j_1}v_n^{j_2})\big\|_{L_t^{\frac32}L_x^{\frac{15}{2}}}\|v_n^{j_3}\|_{X(0,\infty)}\Big)\\
&\lesssim_J\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{j_1,j_2,j_3=1}^J\|v_n^{j_1}v_n^{j_2}\|_{L_t^{\frac32}L_x^{\frac{15}{8}}}=0,\end{aligned}$$

where the last limit is also a consequence of the orthogonality.

For (4.11), note that on $(0,\infty)\times\mathbb{R}^5$

$$\begin{aligned}
&\big\||\nabla|^{\frac12}(F(u_n^J-e^{it\Delta}\omega_n^J)-F(u_n^J))\big\|_{L_t^1L_x^2}\\
\lesssim\ &\big\||\nabla|^{\frac12}[\big(|\nabla|^{-2}(u_n^Je^{it\Delta}\omega_n^J)\big)u_n^J]\big\|_{L_t^1L_x^2}+\big\||\nabla|^{\frac12}[(|\nabla|^{-2}(u_n^Je^{it\Delta}\omega_n^J))e^{it\Delta}\omega_n^J]\big\|_{L_t^1L_x^2}\\
&+\big\||\nabla|^{\frac12}[(|\nabla|^{-2}|u_n^J|^2)e^{it\Delta}\omega_n^J]\big\|_{L_t^1L_x^2}+\big\||\nabla|^{\frac12}[(|\nabla|^{-2}|e^{it\Delta}\omega_n^J|^2)e^{it\Delta}\omega_n^J]\big\|_{L_t^1L_x^2}\\
&+\big\||\nabla|^{\frac12}[(|\nabla|^{-2}|e^{it\Delta}\omega_n^J|^2)u_n^J]\big\|_{L_t^1L_x^2}.
\end{aligned}$$

Using (2.4), Hölder's inequality, the Hardy-Littlewood-Sobolev inequality, the above terms on the right hand side will go to zero as J, n tend to ∞, except

$$\big\||\nabla|^{\frac12}[(|\nabla|^{-2}|u_n^J|^2)e^{it\Delta}\omega_n^J]\big\|_{L_t^1L_x^2((0,\,\infty)\times\mathbb{R}^5)}.$$

By the fractional Leibniz rule and the triangle inequality, it suffices to estimate

$$\big\||\nabla|^{\frac12}(|\nabla|^{-2}|u_n^J|^2)e^{it\Delta}\omega_n^J\big\|_{L_t^1L_x^2((0,\,\infty)\times\mathbb{R}^5)}$$

and

$$\big\|(|\nabla|^{-2}|u_n^J|^2)|\nabla|^{\frac12}e^{it\Delta}\omega_n^J\big\|_{L_t^1L_x^2((0,\,\infty)\times\mathbb{R}^5)}.$$

Using Hölder's, the Hardy-Littlewood-Sobolev inequality, and (2.4), the first integral goes to zero when J, n go to infinity. Then, we are reduced to showing that the second integral has limit zero with J, n.

Replace u_n^J with its definition formula (4.7) to get on $(0,\infty)\times\mathbb{R}^5$

$$\begin{aligned}
&\big\|(|\nabla|^{-2}|u_n^J|^2)|\nabla|^{\frac12}e^{it\Delta}\omega_n^J\big\|_{L_t^1L_x^2}\\
\lesssim&\sum_{j=1}^J\big\|(|\nabla|^{-2}|v_n^j|^2)|\nabla|^{\frac12}e^{it\Delta}\omega_n^J\big\|_{L_t^1L_x^2}+\sum_{j'\neq j}\big\|(|\nabla|^{-2}(v_n^jv_n^{j'}))|\nabla|^{\frac12}e^{it\Delta}\omega_n^J\big\|_{L_t^1L_x^2}\\
&+\sum_{j=1}^J\big\|(|\nabla|^{-2}(v_n^je^{it\Delta}\omega_n^J))|\nabla|^{\frac12}e^{it\Delta}\omega_n^J\big\|_{L_t^1L_x^2}:=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}$$

By (2.5), I_2 will go to zero as J, n go to infinity. Using (2.4), I_3 vanishes as J, n tend to infinity. So, We only need to show that I_1 also vanishes.

For arbitrary $\eta>0$, from (4.9), there exists $J'(\eta)\geq 1$ such that

$$\sum_{j\geq J'}\|v_n^j\|_{S(0,\,\infty)}\leq\eta.$$

Thus, we are reduced to proving that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\big\|(|\nabla|^{-2}|v_n^j|^2)|\nabla|^{\frac12}e^{it\Delta}\omega_n^J\big\|_{L_t^1L_x^2((0,\,\infty)\times\mathbb{R}^5)}=0\quad\text{for all}\quad 1\leq j\leq J'.$$

Fix $1\leq j\leq J'$. A change of variables yields

$$\big\|(|\nabla|^{-2}|v_n^j|^2)|\nabla|^{\frac12}e^{it\Delta}\omega_n^J\big\|_{L_t^1L_x^2}=\big\|\big(|\nabla|^{-2}|v^j|^2\big)|\nabla|^{\frac12}\big[T_{(g_n^j)^{-1}}(e^{it\Delta}\omega_n^J)\big](\cdot-t_n^j)\big\|_{L_t^1L_x^2}.$$

Let $\tilde{\omega}_n^J := [T_{(g_n^j)^{-1}}(e^{it\Delta}\omega_n^J)](\cdot - t_n^j)$, $\mathcal{I} : v^j \mapsto (|\nabla|^{-2}|v^j|^2)$. Note that

$$\|\tilde{\omega}_n^J\|_{S(0,\,\infty)} = \|e^{it\Delta}\omega_n^J\|_{S(0,\infty)}, \quad \|\tilde{\omega}_n^J\|_{X(0,\infty)} = \|e^{it\Delta}\omega_n^J\|_{X(0,\,\infty)}. \tag{4.12}$$

Using Hölder's inequality, the interpolation theorem, we see

$$\begin{aligned}
&\big\|\mathcal{I}(v^j)|\nabla|^{\frac{1}{2}}\tilde{\omega}_n^J\big\|_{L_t^1 L_x^2} \\
\lesssim &\|\mathcal{I}(v^j)\|_{L_t^{12/7} L_x^{15}}\big\||\nabla|^{\frac{1}{2}}\tilde{\omega}_n^J\big\|_{L_t^{12/5} L_x^{30/13}} \\
\lesssim &\|v^j\|_{L_t^{24/7} L_x^{30/7}}\big\|\tilde{\omega}_n^J\big\|_{X(0,\infty)}^{1/2}\big\||\nabla|^{\frac{1}{2}}\tilde{\omega}_n^J\big\|_{L_{t,x}^2}^{1/2}.
\end{aligned}$$

By density, we may assume $\mathcal{I}(v_n^j) \in C_c^\infty(\mathbb{R}\times\mathbb{R}^5)$. It thus suffices to verify

$$\lim_{J\to\infty}\limsup_{n\to\infty}\big\||\nabla|^{\frac{1}{2}}\tilde{\omega}_n^J\big\|_{L_{t,x}^2(K)} = 0$$

for any compact $K \subset \mathbb{R}\times\mathbb{R}^5$. This is a consequence of (2.4) and the following lemma:

Lemma 4.1. *Let $\phi \in \dot{H}_x^{1/2}(\mathbb{R}^5)$. Then*

$$\big\||\nabla|^{\frac{1}{2}}e^{it\Delta}\phi\big\|_{L_{t,x}^2([-T,\,T]\times\{x:\,|x|\le R\})}^2 \lesssim T^{\frac{1}{6}}R^{\frac{5}{3}}\|e^{it\Delta}\phi\|_{L_t^3 L_x^{15/4}}\big\||\nabla|^{\frac{1}{2}}\phi\big\|_{L_x^2}.$$

Proof. The proof is analogous to the one of Lemma 2.5 in [14].

Now, applying perturbation theorem with $\tilde{u} = u_n^J$, $e = (i\partial_t + \Delta)u_n^J - F(u_n^J)$, and using (4.9), we obtain

$$\|u_n^J\|_{S(0,\,\infty)} \lesssim 1 + A_c$$

for all sufficiently large n. This contradicts (4.2), which concludes Step 2.

Combining Step 1 with Step 2, and rearranging the indices, we may find $1 \le J_1 \le J_0$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\|v_n^j\|_{S(0,\,T_{n,j}^+)} = \infty \quad \text{for } 1 \le j \le J_1, \\
&\limsup_{n\to\infty}\|v_n^j\|_{S(0,\,T_{n,j}^+)} < \infty \quad \text{for } j > J_1.
\end{aligned}$$

For $m \in \mathbb{N}$, $n \ge 1$, define an interval K_n^m of the form $[0, \tau]$ by

$$\sup_{1\le j\le J_1}\|v_n^j\|_{S(K_n^m)} = m.$$

Then, v_n^j is defined on K_n^m for all $j \ge 1$ and $\|v_n^j\|_{S(K_n^m)}$ is finite for all $j \ge 1$.

Since u_n^J is a good approximation to u_n, using the same argument as in Step 2, we may obtain

$$\lim_{J\to\infty}\limsup_{n\to\infty}\sup_{t\in K_n^m}\|u_n^J - u_n\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} = 0 \tag{4.13}$$

for each $m \ge 1$.

By the definition of K_n^m, we may choose $1 \leq j_0 = j_0(m,n) \leq J_1$ such that

$$\|v_n^{j_0(m,n)}\|_{S(K_n^m)} = m. \tag{4.14}$$

Moreover, there are infinitely many m satisfying $j_0(m,n) = j_0$ for infinitely many n.

By the definition of A_c, we have

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \sup_{t\in K_n^m} \|v_n^{j_0}\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} \geq A_c. \tag{4.15}$$

Step 3: For all $J \geq 1$ and $m \geq 1$

$$\lim_{n\to\infty} \sup_{t\in K_n^m} \left| \|u_n^J(t)\|_{\dot{H}_x^{1/2}}^2 - \sum_{j=1}^{J} \|v_n^j(t)\|_{\dot{H}_x^{1/2}}^2 - \|\omega_n^J\|_{\dot{H}_x^{1/2}}^2 \right| = 0. \tag{4.16}$$

Proof. Note that for all $J \geq 1$, $m \geq 1$

$$\begin{aligned}
\|u_n^J(t)\|_{\dot{H}_x^{1/2}}^2 &= \big\langle\, |\nabla|^{\frac{1}{2}} u_n^J(t), |\nabla|^{\frac{1}{2}} u_n^J(t) \,\big\rangle \\
&= \sum_{j=1}^{J} \big\| |\nabla|^{\frac{1}{2}} v_n^j \big\|_{\dot{H}_x^{1/2}}^2 + \|\omega_n^J\|_{\dot{H}_x^{1/2}}^2 + \sum_{j'\neq j} \big\langle\, |\nabla|^{\frac{1}{2}} v_n^j(t), |\nabla|^{\frac{1}{2}} v_n^{j'}(t) \,\big\rangle \\
&\quad + \sum_{j=1}^{J} \Big(\big\langle\, |\nabla|^{\frac{1}{2}} e^{it\Delta}\omega_n^J, |\nabla|^{\frac{1}{2}} v_n^j(t) \,\big\rangle + \big\langle\, |\nabla|^{\frac{1}{2}} v_n^j(t), |\nabla|^{\frac{1}{2}} e^{it\Delta}\omega_n^J \,\big\rangle \Big).
\end{aligned}$$

Thus, to establish (4.16), it suffices to show that for all $t_n \in K_n^m$,

$$\lim_{n\to\infty} \big\langle\, |\nabla|^{\frac{1}{2}} v_n^j(t_n)\,,\, |\nabla|^{\frac{1}{2}} v_n^{j'}(t_n) \,\big\rangle = 0 \tag{4.17}$$

and

$$\lim_{n\to\infty} \big\langle\, |\nabla|^{\frac{1}{2}} e^{it_n\Delta}\omega_n^J\,,\, |\nabla|^{\frac{1}{2}} v_n^j(t_n) \,\big\rangle = 0 \tag{4.18}$$

for all $1 \leq j$, $j' \leq J$, $j \neq j'$.

We only deal with (4.18), as (4.17) can be done in the same manner, using (2.5).

Do a change of variables, the formula in (4.18) becomes

$$\big\langle\, |\nabla|^{\frac{1}{2}} e^{it_n(\lambda_n^j)^{-2}\Delta}[(g_n^j)^{-1}\omega_n^J]\,,\, |\nabla|^{\frac{1}{2}} v^j(t_n^j + t_n(\lambda_n^j)^{-2}) \,\big\rangle. \tag{4.19}$$

Since $t_n \in K_n^m \subset [0, T_{n,j}^+)$ for all $1 \leq j \leq J_1$ and v_j has maximal-life span $I^j = \mathbb{R}$ for $j > J_1$, we have $t_n(\lambda_n^j)^{-2} + t_n^j \in I^j$ for all $j \geq 1$. By passing to a subsequence in n, we may assume $t_n(\lambda_n^j)^{-2} + t_n^j \to \tau^j$.

If τ^j is finite, then by the continuity of the flow, $v^j(t_n(\lambda_n^j)^{-2} + t_n^j) \to v^j(\tau^j)$ in $\dot{H}_x^{1/2}$.

From (2.6), we have

$$\lim_{n\to\infty} \big\| e^{it_n(\lambda_n^j)^{-2}\Delta}[(g_n^j)^{-1}\omega_n^J] \big\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} = \lim_{n\to\infty} \|\omega_n^J\|_{\dot{H}_x^{1/2}} \lesssim A_c.$$

Combining this with (4.19), and using Lemma 2.10, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{it_n\Delta}\omega_n^J \,,\, |\nabla|^{\frac12} v_n^j(t_n^j)\,\big\rangle \\
&= \lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{it_n(\lambda_n^j)^{-2}\Delta}[(g_n^j)^{-1}\omega_n^J]\,,\, |\nabla|^{\frac12} v^j(\tau^j)\,\big\rangle \\
&= \lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{-it_n^j\Delta}[(g_n^j)^{-1}\omega_n^J]\,,\, |\nabla|^{\frac12} e^{-i\tau^j\Delta} v^j(\tau^j)\,\big\rangle \\
&= 0,
\end{aligned}$$

which concludes (4.16).

If $\tau^j = +\infty$, then since $t_n(\lambda_n^j)^{-2} \geq 0$, we must have $\sup I^j = \infty$ and v^j scatters forward in time. Therefore, there exists $\psi^j \in \dot{H}_x^{1/2}(\mathbb{R}^5)$ such that

$$\lim_{n\to\infty} \big\| v^j(t_n^j + t_n(\lambda_n^j)^2) - e^{i(t_n(\lambda_n^j)^{-2}+t_n^j)\Delta}\psi^j \big\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} = 0.$$

Thus, together with (4.19) and Lemma 2.10 yields

$$\begin{aligned}
&\lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{it_n\Delta}\omega_n^J \,,\, |\nabla|^{\frac12} v_n^j(t_n^j)\,\big\rangle \\
&= \lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{it_n(\lambda_n^j)^{-2}\Delta}[(g_n^j)^{-1}\omega_n^J]\,,\, e^{i(t_n(\lambda_n^j)^{-2}+t_n^j)\Delta}|\nabla|^{\frac12}\psi^j\,\big\rangle \\
&= \lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{it_n^j\Delta}[(g_n^j)^{-1}\omega_n^J],\, |\nabla|^{\frac12}\psi^j\,\big\rangle \\
&= 0.
\end{aligned}$$

If $\tau^j = -\infty$, then we must have $t_n^j \to -\infty$ as $n \to \infty$. Indeed, since $t_n(\lambda_n^j)^{-2} \geq 0$ and $\inf I^j < \infty$, t_n^j can not converges to $+\infty$; if $t_n^j \equiv 0$, then since $\inf I^j < 0$, we have $t_n(\lambda_n^j)^{-2} \leq 0$, which contradicts $t_n \in K_n^m \subset [0, T_{n,j}^+)$. Hence, $\inf I^j = -\infty$. By the definition of nonlinear profile, v^j scatters backward in time to $e^{it\Delta}\phi^j$.

$$\lim_{n\to\infty} \big\| v^j(t_n^j + t_n(\lambda_n^j)^2) - e^{i(t_n(\lambda_n^j)^{-2}+t_n^j)\Delta}\phi^j \big\|_{\dot{H}_x^{1/2}(\mathbb{R}^5)} = 0.$$

Combining this with (4.19) gives

$$\begin{aligned}
&\lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{it_n\Delta}\omega_n^J \,,\, |\nabla|^{\frac12} v_n^j(t_n^j)\,\big\rangle \\
&= \lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{it_n(\lambda_n^j)^{-2}\Delta}[(g_n^j)^{-1}\omega_n^J]\,,\, e^{i(t_n(\lambda_n^j)^{-2}+t_n^j)\Delta}|\nabla|^{\frac12}\phi^j\,\big\rangle \\
&= \lim_{n\to\infty} \big\langle\, |\nabla|^{\frac12} e^{it_n^j\Delta}[(g_n^j)^{-1}\omega_n^J]\,,\, |\nabla|^{\frac12}\phi^j\,\big\rangle \\
&= 0.
\end{aligned}$$

This completes the proof of Step 3.

From (4.1), (4.13), (4.16)

$$A_c^2 \geq \limsup_{n\to\infty} \sup_{t\in K_n^m} \|u_n(t)\|_{\dot{H}_x^{1/2}}^2 \geq \lim_{n\to\infty} \sup_{t\in K_n^m} \Big(\sum_{j=1}^J \|v_n^j\|_{\dot{H}^{1/2}}^2 + \|\omega_n^J\|_{\dot{H}^{1/2}}^2 \Big).$$

Invoking (4.15) that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\sup_{t\in K_n^m}\|v_n^{j_0}(t)\|_{\dot H_x^{1/2}}\geq A_c,$$

we conclude that $v_n^j\equiv 0$ for all $j\neq j_0$, and $\omega_n^{j_0}\to 0$ in $\dot H_x^{1/2}(\mathbb{R}^5)$. Thus,

$$u_n(0)=g_ne^{i\tau_n\Delta}\phi+\omega_n \tag{4.20}$$

for some $g_n\in G_{rad}$, $\tau_n\in\mathbb{R}$, ϕ, $\omega_n\in\dot H_x^{1/2}(\mathbb{R}^5)$ with $\omega_n\to 0$ in $\dot H^{1/2}$. We also have $\tau_n\equiv 0$ or $\tau_n\to\pm\infty$.

If $\tau_n\equiv 0$, then $u_n(0)\to\phi$ in $\dot H_x^{1/2}$ modulo scaling. This proves Proposition 4.1.

If $\tau_n\to\pm\infty$, by time-reversal symmetry, we only consider $\tau_n\to+\infty$. In this case, by the Strichartz estimate, we have $\|e^{it\Delta}\phi\|_{S(\mathbb{R}^+)}<\infty$. By a change of variables,

$$\lim_{n\to\infty}\|e^{it\Delta}e^{-i\tau_n\Delta}\phi\|_{S(\mathbb{R}^+)}=0.$$

Taking the group action yields

$$\lim_{n\to\infty}\|e^{it\Delta}g_ne^{-i\tau_n\Delta}\phi\|_{S(\mathbb{R}^+)}=0.$$

From (4.20), (2.4), we deduce

$$\lim_{n\to\infty}\|e^{it\Delta}u_n(0)\|_{S(\mathbb{R}^+)}=0.$$

Invoking perturbation theorem, we obtain

$$\lim_{n\to\infty}\|u_n\|_{S(\mathbb{R}^+)}=0,$$

which contradicts (4.2). This completes the proof of Proposition 4.1. □

4.2 Proof of Theorem 1.3

Proof. Suppose Theorem 1.2 failed. Then $A_c<\frac{\sqrt{6}}{3}\|Q\|_{\dot H_x^{1/2}}$, and by the definition of A_c, we can find a sequence of solutions $u_n: I_n\times\mathbb{R}^5\mapsto\mathbb{C}$ to (1.1) with I_n compact,

$$\sup_{n\geq 1}\sup_{t\in I_n}\big\||\nabla|^{\frac12}u_n(t)\big\|_2=A_c,\quad \lim_{n\to\infty}\|u_n\|_{S(I_n)}=\infty. \tag{4.21}$$

Then exists $t_n\in I_n$ such that

$$\lim_{n\to\infty}\|u_n\|_{S(-\infty,\,t_n)}=\lim_{n\to\infty}\|u_n\|_{S(t_n,\,\infty)}=\infty. \tag{4.22}$$

By time-translation symmetry, we set all $t_n=0$. Applying Proposition 4.1, there exists (up to a subsequence) $g_n\in G_{rad}$ and a function $u_0\in\dot H_x^{1/2}(\mathbb{R}^5)$ such that $g_nu_n(0)\to u_0$ in $\dot H_x^{1/2}$. By taking group action T_{g_n} to the solution u_n, we may make g_n be the identity. Thus $u_n(0)\to u_0$ in $\dot H_x^{1/2}$.

Let $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ be the maximal-life span solution to (1.1) with initial data $u(0) = u_0$. Then, Theorem 1.1 implies $I \subseteq \liminf I_n$ and

$$\lim_{n\to\infty} \sup_{t\in K} \|u_n(t) - u(t)\|_{\dot{H}_x^{1/2}} = 0$$

for all compact $K \subset I$.

Thus, from (4.21)

$$\sup_{t\in I} \|u(t)\|_{\dot{H}_x^{1/2}} \leq A_c. \tag{4.23}$$

On the other hand, we claim that u blows up both froward and backward in time. If not, $\|u\|_{S(0,\,\infty)} < \infty$, $\|u\|_{S(-\infty,\,0)} < \infty$. From perturbation theorem, $\|u_n\|_{S(0,\,\infty)} < \infty$, $\|u_n\|_{S(-\infty,\,0)} < \infty$ for n large enough, which contradicts (4.22).

So, by the definition of A_c

$$\sup_{t\in I} \|u(t)\|_{\dot{H}_x^{1/2}} \geq A_c$$

which together with (4.23) yields

$$\sup_{t\in I} \|u(t)\|_{\dot{H}_x^{1/2}} = A_c.$$

Next, we prove that u is almost periodic modulo scaling. For arbitrary sequence $\tau_n \in I$, we have

$$\|u\|_{S(-\infty,\,\tau_n)} = \|u\|_{S(\tau_n,\,\infty)} = \infty,$$

since u blows up both forward and backward. From Proposition 4.1, $u(\tau_n)$ has a subsequence which converges in $\dot{H}_x^{1/2}(\mathbb{R}^5)$ modulo scaling. Thus $\{u(t) : t \in I\}$ is precompact in $\dot{H}_x^{1/2}(\mathbb{R}^5)$ modulo G_{rad}(Remark 1.2). This completes the proof of the first part of Theorem 1.3.

An almost periodic blowup solution which obeys the three scenarios in Theorem 1.3 can be extracted from the above solution by renormalization and subsequential limits. As we've pointed out, the process is similar to that in [13], [14], and we refer the readers to these papers for a detailed discussion.

5 No finite-time blow up

In this section, we prove

Theorem 5.1. *There exists no such maximal life-span solution $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ to* (1.1) *that is almost periodic modulo scaling and*

$$\sup_{t\in I} \|u(t)\|_{\dot{H}_x^{1/2}} < \frac{\sqrt{6}}{3} \|Q\|_{\dot{H}_x^{1/2}}, \quad \|u\|_{S(I)} = \infty \tag{5.1}$$

and either $|\inf I| < \infty$ or $\sup I < \infty$.

Proof. Assume for a contradiction that there existed such a solution. Without loss of generality, we may assume $\sup I < \infty$. We claim that

$$\liminf_{t \nearrow \sup I} N(t) = \infty. \tag{5.2}$$

If not, we may find a time sequence $t_n \in I$ such that $t_n \nearrow \sup I$, $\liminf_n N(t_n) < \infty$. For each $n \geq 1$, define $v_n : I_n \times \mathbb{R}^5 \mapsto \mathbb{C}$ by

$$v_n(t,x) := u(t_n + tN(t_n)^{-2},\ xN(t_n)^{-1})$$

with $I_n := \{\, t \in \mathbb{R} : t_n + tN(t_n)^{-2} \in I \,\}$. Then v_n is also a solution to (1.1), $\{v_n(0)\}$ is precompact in $\dot{H}_x^{1/2}(\mathbb{R}^5)$. After passing to a subsequence, we may assume $v_n(0) \to v_0$ in $\dot{H}_x^{1/2}(\mathbb{R}^5)$. Since $\|v_n(0)\|_{\dot{H}_x^{1/2}} = \|u(t_n)\|_{\dot{H}_x^{1/2}}$, v_0 is not identically zero.

Let v be the maximal life-span solution to (1.1) with initial data v_0, and maximal life-span $(-T_-,\ T_+)$, $-\infty \leq T_- < 0 < T_+ \leq \infty$. For any compact $J \subset (-T_-,\ T_+)$, from local wellposedness theory, for sufficiently large n, v_n is wellposed on J and $\|v_n\|_{S(J)} < \infty$. Thus, u is wellposed on the interval $J_n = \{\, t_n + tN(t_n)^{-2} : t \in J \,\}$ and $\|u\|_{S(J_n)} < \infty$. But $\liminf_{t \nearrow \sup I} N(t) < \infty$ implies that $\|u\|_S$ is finite beyond $\sup I$, which contradicts the assumption that u blows up on I.

Next, we will prove that for all $R > 0$

$$\limsup_{t \nearrow \sup I} \int_{|x| \leq R} |u(t,x)|^2 \,\mathrm{d}x = 0. \tag{5.3}$$

Let $\eta > 0$, $t \in I$. Using Hölder's inequality, Sobolev's embedding theorem, (5.2)

$$\begin{aligned}
\int_{|x| \leq R} |u(t,x)|^2 \,\mathrm{d}x &\leq \int_{|x| \leq \eta R} |u(t,x)|^2 \,\mathrm{d}x + \int_{\eta R \leq |x| \leq R} |u(t,x)|^2 \,\mathrm{d}x \\
&\leq \eta R \left(\int |u(t,x)|^{5/2} \,\mathrm{d}x \right)^{4/5} + R \left(\int_{|x| \geq \eta R} |u(t,x)|^{5/2} \,\mathrm{d}x \right)^{4/5} \\
&\lesssim \eta R \|u(t)\|_{\dot{H}_x^{1/2}}^2 + R \left(\int_{|x| \geq \eta R} |u(t,x)|^{5/2} \,\mathrm{d}x \right)^{4/5}.
\end{aligned}$$

The first term will go to zero as η tends to zero. On the other hand, from (5.2), almost periodic modulo scaling, and (1.5), we have

$$\limsup_{t \nearrow \sup I} \int_{|x| \geq R} |u(t,x)|^{5/2} \,\mathrm{d}x \leq \limsup_{t \nearrow \sup I} \int_{|x| \geq C(\eta)/N(t)} |u(t,x)|^{5/2} \,\mathrm{d}x = 0.$$

Thus (5.3) holds.

We will prove from (5.3) that u is identically zero.

For $t \in I$, define

$$M_R(t) := \int_{\mathbb{R}^5} \phi\Big(\frac{|x|}{R}\Big) |u(t,x)|^2 \,\mathrm{d}x$$

where ϕ is a smooth, radial function with

$$\phi(r) = \begin{cases} 1, & r \le 1 \\ 0, & r \ge 2. \end{cases}$$

By (5.3),

$$\limsup_{t \nearrow \sup I} M_R(t) = 0 \quad \text{for all} \quad R > 0. \tag{5.4}$$

A direct computation involving Plancherel, Hardy's inequality and (5.1) yields

$$\begin{aligned}
\big|\partial_t M_R(t)\big| \quad &\lesssim \quad \int_{\mathbb{R}^5} \Big(\Big|\frac{x}{R^2}\phi'\big(\frac{|x|}{R}\big)\bar{u}\Big|\Big)^{\widehat{}}(\xi)|\xi||\hat{u}|\,\mathrm{d}\xi \\
&\lesssim \quad \Big\||\nabla|^{\frac12}\big(\frac{x}{R^2}\phi'\big(\frac{|x|}{R}\big)\bar{u}\big)\Big\|_2 \||\xi|^{\frac12}\hat{u}\|_2 \\
&\lesssim_u \quad \Big\||\nabla|^{\frac12}\big(\frac{x}{R^2}\phi'\big(\frac{|x|}{R}\big)\big)\bar{u}\Big\|_2 + \Big\|\frac{x}{R^2}\phi'\big(\frac{|x|}{R}\big)|\nabla|^{\frac12}\bar{u}\Big\|_2 \\
&\lesssim_u \quad \Big\||x|^{\frac12}|\nabla|^{\frac12}\big(\frac{x}{R^2}\phi'\big(\frac{|x|}{R}\big)\big)\Big\|_{L^\infty}\Big\|\frac{\bar{u}}{|x|^{1/2}}\Big\|_2 + \Big\|\frac{x}{R^2}\phi'\big(\frac{|x|}{R}\big)\Big\|_{L^\infty}\||\nabla|^{\frac12}u\|_2. \\
&\lesssim_u \quad \Big\||x|^{\frac12}|\nabla|^{\frac12}\big(\frac{x}{R^2}\phi'\big(\frac{|x|}{R}\big)\big)\Big\|_{L^\infty} + \frac{1}{R}.
\end{aligned}$$

Furthermore, if $|x| \le 4R$, then by our chosen of ϕ

$$\Big\||x|^{\frac12}|\nabla|^{\frac12}\Big(\frac{x}{R^2}\phi'\big(\frac{|x|}{R}\big)\Big)\Big\|_{L^\infty} \lesssim \frac{1}{R}.$$

If $|x| > 4R$, then using the intrinsic description of derivatives, we have the following

$$\begin{aligned}
\frac{|x|^{\frac12}}{R^2}|\nabla|^{\frac12}\Big(x\phi'\big(\frac{|x|}{R}\big)\Big) = \frac{1}{R^2}\int_{\mathbb{R}^5} &\frac{|x|^{\frac12}\big[x\phi'\big(\frac{|x|}{R}\big) - y\phi'\big(\frac{|y|}{R}\big)\big]}{|x-y|^{5+\frac12}}\,\mathrm{d}y \\
=&\frac{1}{R^2}\int_{|x-y|\ge\frac12|x|} \frac{|x|^{\frac12}\big[x\phi'\big(\frac{|x|}{R}\big) - y\phi'\big(\frac{|y|}{R}\big)\big]}{|x-y|^{5+\frac12}}\,\mathrm{d}y \\
&+ \frac{1}{R^2}\int_{|x-y|<\frac12|x|} \frac{|x|^{\frac12}\big[x\phi'\big(\frac{|x|}{R}\big) - y\phi'\big(\frac{|y|}{R}\big)\big]}{|x-y|^{5+\frac12}}\,\mathrm{d}y.
\end{aligned}$$

It is easily to check that the first integration has a bound R^{-1}, since $|x-y| \ge \frac12|x| \ge 2R$. For the second one, we have $|y| > |x| - |x-y| > \frac12|x| > 2R$, and by the property of ϕ, it follows that the integration is equal to zero.

From the above, we obtain

$$\big|\partial_t M_R(t)\big| \lesssim_u \frac{1}{R}.$$

Thus, by the Fundamental Theorem of Calculus

$$M_R(t_1) \lesssim M_R(t_2) + \int_{t_2}^{t_1} \partial_t M_R(t)\,\mathrm{d}t \lesssim M_R(t_2) + \frac{1}{R}|t_1 - t_2|$$

for all $t_1, t_2 \in I$ and $R > 0$.

Let $t_2 \nearrow \sup I$ and from (5.4), we obtain

$$M_R(t_1) \lesssim_u \frac{1}{R} |\sup I - t_1|.$$

Let $R \to \infty$, then we deduce that $M(u(t)) = 0$ for all $t \in I$. This implies that $u \equiv 0$, which contradicts $\|u\|_{S(I)} = \infty$. This completes the proof of Theorem 5.1.

6 Negative regularity

In this section, we prove the following

Theorem 6.1 (Negative regularity in the global case)**.** *Let u be a global radially symmetric solution to* (1.1) *which is almost periodic modulo scaling. Suppose also that*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{\dot{H}_x^{1/2}} < \frac{\sqrt{6}}{3} \|Q\|_{\dot{H}_x^{1/2}} \tag{6.1}$$

and

$$\inf_{t\in\mathbb{R}} N(t) \gtrsim 1. \tag{6.2}$$

Then, $u \in L_t^\infty \dot{H}^{-\varepsilon}(\mathbb{R} \times \mathbb{R}^5)$ for some $\varepsilon > 0$. In particular, $u \in L_t^\infty L_x^2$.

In order to prove Theorem 6.1, we first establish a recurrence formula.

Given $\eta > 0$, from Remark 1.1, there exists $N_0 = N_0(\eta)$ such that

$$\|u_{\le N_0}(t)\|_{\dot{H}_x^{1/2}} \le \eta. \tag{6.3}$$

Now, define

$$A(N) := N^{-\frac{3}{4}} \sup_{t\in\mathbb{R}} \|u_N(t)\|_{L_x^4}$$

for all $N \le 8N_0$.

Note that by Bernstein's inequality, Sobolev's embedding theorem

$$A(N) \lesssim N^{-\frac{3}{4}} N^{\frac{3}{4}} \|u_N\|_{L_t^\infty L_x^{5/2}} \le \|u\|_{L_t^\infty \dot{H}_x^{1/2}} < \infty.$$

Moreover, $A(N)$ satisfies the following recurrence formula

Lemma 6.1. *For $N \le 8N_0$*

$$A(N) \lesssim_u \left(\frac{N}{N_0}\right)^{\frac{1}{2}} + \eta^2 \sum_{8N \le N_1 \le N_0} \left(\frac{N}{N_1}\right)^{\frac{1}{8}} A(N_1) + \eta^2 \sum_{N_1 \le 8N} \left(\frac{N_1}{N}\right)^{\frac{3}{4}} A(N_1). \tag{6.4}$$

Proof. We only need to prove that for all $t \in \mathbb{R}$

$$N^{-\frac{3}{4}}\|u_N(t)\|_{L_x^4} \lesssim \text{ RHS of (6.4)}.$$

By the time-translation symmetry, it reduces to prove

$$N^{-\frac{3}{4}}\|u_N(0)\|_{L_x^4} \lesssim \text{ RHS of (6.4)}.$$

By the Duhamel formula (1.7), the triangle, Bernstein's and the dispersive inequality, we have

$$\begin{aligned}
N^{-\frac{3}{4}}\|u_N(0)\|_{L_x^4} \leq & N^{-\frac{3}{4}}\Big\|\int_0^{N^{-2}} e^{-it\Delta}P_N F(u(t))\,\mathrm{d}t\Big\|_{L_x^4} \\
& + N^{-\frac{3}{4}}\Big\|\int_{N^{-2}}^{\infty} e^{-it\Delta}P_N F(u(t))\,\mathrm{d}t\Big\|_{L_x^4} \\
\lesssim & N^{\frac{1}{2}}\Big\|\int_0^{N^{-2}} e^{-it\Delta}P_N F(u(t))\,\mathrm{d}t\Big\|_{L_x^2} \\
& + N^{-\frac{3}{4}}\|P_N F(u)\|_{L_t^\infty L_x^{4/3}}\int_{N^{-2}}^{\infty} t^{-\frac{5}{4}}\,\mathrm{d}t \\
\lesssim & N^{-\frac{3}{2}}\|P_N F(u)\|_{L_t^\infty L_x^2} + N^{-\frac{1}{4}}\|P_N F(u)\|_{L_t^\infty L_x^{4/3}} \\
\lesssim & N^{-\frac{1}{4}}\|P_N F(u)\|_{L_t^\infty L_x^{4/3}}.
\end{aligned}$$

Decompose u as

$$u := u_{\geq N_0} + u_{\frac{N}{8}\leq \cdot < N_0} + u_{<\frac{N}{8}},$$

and then make a corresponding expansion of $F(u)$, we obtain terms constitute $F(u)$ of the following types

1. At least one high frequency, i.e. $|\nabla|^{-2}(uu_{\geq N_0})u$, or $|\nabla|^{-2}(u^2)u_{\geq N_0}$;
2. Non-high frequency component and at least one lower frequency:

$$|\nabla|^{-2}(u_{<\frac{N}{8}}u_{\leq N_0})u_{\leq N_0}, \quad |\nabla|^{-2}(u_{\leq N_0}^2)u_{<\frac{N}{8}};$$

3. All medium components: $|\nabla|^{-2}(u_{\frac{N}{8}\leq \cdot < N_0}^2)u_{\frac{N}{8}\leq \cdot < N_0}$.

Case 1(**At least one high frequency**). Using Bernstein's inequality, discarding the projector P_N, and then using the Hardy-Littlewood-Sobolev, Hölder's and Bernstein's inequality, Sobolev embedding, we have

$$\begin{aligned}
N^{-\frac{1}{4}}\big\|P_N(|\nabla|^{-2}(uu_{\geq N_0})u)\big\|_{L_t^\infty L_x^{4/3}} \lesssim & N^{\frac{1}{2}}\big\||\nabla|^{-2}(uu_{\geq N_0})u\big\|_{L_t^\infty L_x^{10/9}} \\
\lesssim_u & N^{\frac{1}{2}}\big\||\nabla|^{-2}(uu_{\geq N_0})\|_{L_t^\infty L_x^2}\|u\|_{L_t^\infty L_x^{5/2}} \\
\lesssim_u & N^{\frac{1}{2}}\|uu_{\geq N_0}\|_{L_t^\infty L_x^{10/9}} \\
\lesssim_u & N^{\frac{1}{2}}\|u\|_{L_t^\infty L_x^{5/2}}\|u_{\geq N_0}\|_{L_t^\infty L_x^2} \\
\lesssim_u & N^{\frac{1}{2}}N_0^{-\frac{1}{2}},
\end{aligned}$$

$$
\begin{aligned}
N^{-\frac{1}{4}}\big\|P_N(|\nabla|^{-2}(u^2)u_{\geq N_0})\big\|_{L_t^\infty L_x^{4/3}} &\lesssim N^{\frac{1}{2}}\big\||\nabla|^{-2}(u^2)u_{\geq N_0}\big\|_{L_t^\infty L_x^{10/9}}\\
&\lesssim N^{\frac{1}{2}}\big\||\nabla|^{-2}(u^2)\big\|_{L_t^\infty L_x^{5/2}}\|u_{\geq N_0}\|_{L_t^\infty L_x^2}\\
&\lesssim N^{\frac{1}{2}}\|u\|^2_{L_t^\infty L_x^{5/2}}\|u_{\geq N_0}\|_{L_t^\infty L_x^2}\\
&\lesssim_u N^{\frac{1}{2}}N_0^{-\frac{1}{2}};
\end{aligned}
$$

Case 2(**Lower frequency components**). By the triangle, Bernstein's inequality, Sobolev's embedding theorem, Hölder's and the Hardy-Littlewood-Sobolev inequality

$$
\begin{aligned}
&N^{-\frac{1}{4}}\big\|P_N(|\nabla|^{-2}(u_{<\frac{N}{8}}u_{\leq N_0})u_{\leq N_0})\big\|_{L_t^\infty L_x^{4/3}}\\
\lesssim\ & N^{-\frac{1}{4}}\big\|P_{>\frac{N}{8}}\big(|\nabla|^{-2}(u_{<\frac{N}{8}}u_{\leq N_0})\big)u_{\leq N_0}\big\|_{L_t^\infty L^{4/3}}\\
&+N^{-\frac{1}{4}}\big\||\nabla|^{-2}(u_{<\frac{N}{8}}u_{\leq N_0})P_{>\frac{N}{8}}u_{\leq N_0}\big\|_{L_t^\infty L^{4/3}}\\
\lesssim\ & N^{-\frac{1}{4}}\big\|P_{>\frac{N}{8}}|\nabla|^{-2}(u_{<\frac{N}{8}}u_{\leq N_0})\big\|_{L_t^\infty L_x^{20/7}}\|u_{\leq N_0}\|_{L_t^\infty L_x^{5/2}}\\
&+N^{-\frac{1}{4}}\big\||\nabla|^{-2}(u_{<\frac{N}{8}}u_{\leq N_0})\big\|_{L_t^\infty L_x^4}\|P_{>\frac{N}{8}}u_{\leq N_0}\|_{L_t^\infty L_x^2}\\
\lesssim\ & \eta N^{-\frac{3}{4}}\big\|u_{<\frac{N}{8}}u_{\leq N_0}\big\|_{L_t^\infty L_x^{20/13}}+N^{-\frac{3}{4}}\big\|u_{<\frac{N}{8}}u_{\leq N_0}\big\|_{L_t^\infty L_x^{20/13}}\big\||\nabla|^{\frac{1}{2}}u_{\leq N_0}\big\|_{L_t^\infty L_x^2}\\
\lesssim_u\ & \eta^2\sum_{N_1\leq\frac{N}{8}}\left(\frac{N_1}{N}\right)^{\frac{3}{4}}A(N_1),
\end{aligned}
$$

$$
\begin{aligned}
&N^{-\frac{1}{4}}\big\|P_N\big(|\nabla|^{-2}(u^2_{\leq N_0})u_{<\frac{N}{8}}\big)\big\|_{L_t^\infty L_x^{4/3}}\leq N^{-\frac{1}{4}}\big\|P_{>\frac{N}{4}}|\nabla|^{-2}(u^2_{\leq N_0})u_{<\frac{N}{8}}\big\|_{L_t^\infty L_x^{4/3}}\\
\lesssim\ & N^{-\frac{3}{4}}\big\||\nabla|^{-\frac{3}{2}}(u^2_{\leq N_0})\big\|_{L_t^\infty L_x^2}\|u_{<\frac{N}{8}}\|_{L_t^\infty L_x^4}\lesssim N^{-\frac{3}{4}}\|u^2_{\leq N_0}\|_{L_t^\infty L_x^{5/4}}\|u_{<\frac{N}{8}}\|_{L_t^\infty L_x^4}\\
\lesssim\ & \eta^2\sum_{N_1\leq\frac{N}{8}}\left(\frac{N_1}{N}\right)^{\frac{3}{4}}A(N_1);
\end{aligned}
$$

Case 3(**Medium components**). By Bernstein's, the Hardy-Littlewood-Sobolev, the triangle and Hölder's inequality

$$
\begin{aligned}
&N^{-\frac{1}{4}}\big\|P_N(|\nabla|^{-2}(u^2_{\frac{N}{8}\leq\cdot<N_0})u_{\frac{N}{8}\leq\cdot<N_0})\big\|_{L_t^\infty L_x^{4/3}}\\
\lesssim\ & N^{\frac{1}{8}}\big\||\nabla|^{-2}(u^2_{\frac{N}{8}\leq\cdot<N_0})u_{\frac{N}{8}\leq\cdot<N_0}\big\|_{L_t^\infty L_x^{40/33}}\\
\lesssim\ & \sum_{\frac{N}{8}\leq N_1\leq N_2,\,N_3\leq N_0}N^{\frac{1}{8}}\big\||\nabla|^{-2}(u_{N_1}u_{N_2})u_{N_3}\big\|_{L_t^\infty L_x^{40/33}}\\
&+\sum_{\frac{N}{8}\leq N_3\leq N_1\leq N_2\leq N_0}N^{\frac{1}{8}}\big\||\nabla|^{-2}(u_{N_1}u_{N_2})u_{N_3}\big\|_{L_t^\infty L_x^{40/33}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{\frac{N}{8}\le N_1\le N_2, N_3\le N_0} N^{\frac18}\big\||\nabla|^{-2}(u_{N_1}u_{N_2})\big\|_{L_t^\infty L_x^{40/13}}\big\|u_{N_3}\big\|_{L_t^\infty L_x^2}\\
&\quad+\sum_{\frac{N}{8}\le N_3\le N_1\le N_2\le N_0} N^{\frac18}\big\||\nabla|^{-2}(u_{N_1}u_{N_2})\big\|_{L_t^\infty L_x^{40/23}}\big\|u_{N_3}\big\|_{L_t^\infty L_x^4}\\
&\lesssim_u \sum_{\frac{N}{8}\le N_1\le N_2, N_3\le N_0} N^{\frac18}\|u_{N_1}u_{N_2}\|_{L_t^\infty L_x^{40/29}} N_3^{-\frac12}\\
&\quad+\sum_{\frac{N}{8}\le N_3\le N_1\le N_2\le N_0} N^{\frac18}\|u_{N_1}u_{N_2}\|_{L_t^\infty L_x^{40/39}}\|u_{N_3}\|_{L_t^\infty L_x^4}\\
&\lesssim_u \sum_{\frac{N}{8}\le N_1\le N_2, N_3\le N_0} N^{\frac18}\|u_{N_1}\|_{L_t^\infty L_x^4}\|u_{N_2}\|_{L_t^\infty L_x^{40/19}} N_3^{-\frac12}\\
&\quad+\sum_{\frac{N}{8}\le N_3\le N_1\le N_2\le N_0} N^{\frac18}\|u_{N_1}\|_{L_t^\infty L_x^2}\|u_{N_2}\|_{L_t^\infty L_x^{40/19}}\|u_{N_3}\|_{L_t^\infty L_x^4}\\
&\lesssim_u \eta^2 \sum_{\frac{N}{8}\le N_1\le N_2, N_3\le N_0} N^{\frac18}N_2^{-\frac38}N_3^{-\frac12}\|u_{N_1}\|_{L_t^\infty L_x^4}\\
&\quad+\eta^2\sum_{\frac{N}{8}\le N_3\le N_1\le N_2\le N_0} N^{\frac18}N_1^{-\frac12}N_2^{-\frac38}\|u_{N_3}\|_{L_t^\infty L_x^4}\\
&\lesssim_u \eta^2\sum_{\frac{N}{8}\le N_1\le N_0}\left(\frac{N}{N_1}\right)^{\frac18} A(N_1).
\end{aligned}
$$

This concludes the proof of Lemma 6.1.

Proposition 6.1. *Let u be as in Theorem 6.1. Then*

$$u\in L_t^\infty L_x^p \quad \textit{for } \frac{22}{9}\le p<\frac52,$$

Furthermore, by the Hardy-Littlewood-Sobolev inequality

$$|\nabla|^{\frac12}F(u)\in L_t^\infty L_x^r \quad \textit{for } \frac{110}{101}\le r<\frac{10}{9}.$$

Proof. Let $N=8{\cdot}2^{-k}N_0$, applying Lemma 2.1 with $b_k=(8{\cdot}2^{-k})^{\frac18}$, $x_k=A(8{\cdot}2^{-k}N_0)$, we obtain

$$\|u_N\|_{L_t^\infty L_x^4}\lesssim_u N^{7/8+} \quad \text{for all} \quad N\le 8N_0.$$

By the interpolation theorem, Bernstein's inequality, and (6.1)

$$
\begin{aligned}
\|u_N\|_{L_t^\infty L_x^p} &\lesssim \|u_N\|_{L_t^\infty L_x^4}^{2-\frac4p}\|u_N\|_{L_t^\infty L_x^2}^{\frac4p-1}\\
&\lesssim_u N^{\frac78(2-\frac4p)+}N^{-\frac12(\frac4p-1)}\\
&\lesssim_u N^{\frac94-\frac{11}{2p}+}
\end{aligned}
$$

for all $N \leq 8N_0$.

Thus, using Bernstein's inequality together with (6.1), we have

$$\|u\|_{L_t^\infty L_x^p} \leq \|u_{\leq N_0}\|_{L_t^\infty L_x^p} + \|u_{>N_0}\|_{L_t^\infty L_x^p} \lesssim_u \sum_{N \leq N_0} N^{\frac{9}{4}-\frac{11}{2p}+} + \sum_{N>N_0} N^{2-\frac{5}{p}} \lesssim_u 1.$$

Proposition 6.2 (Some negative regularity)**.** *Let u be as in Theorem 6.1. Assume also that $|\nabla|^s F(u) \in L_t^\infty L_x^r$ for some $\frac{110}{101} \leq r < \frac{10}{9}$ and some $0 \leq s \leq \frac{1}{2}$. Then there exists $s_0 = s_0(r) > 0$ such that $u \in L_t^\infty \dot{H}_x^{s-s_0+}$.*

Proof. It only needs to prove that

$$\big\||\nabla|^s u_N\big\|_{L_t^\infty L_x^2} \lesssim N^{s_0} \quad \text{for all} \quad N > 0,\ s_0 := \frac{5}{r} - \frac{9}{2}. \tag{6.5}$$

In fact, by Bernstein's inequality and (6.1)

$$\begin{aligned}
\big\||\nabla|^{s-s_0+}u\big\|_{L_t^\infty L_x^2} &\leq \big\||\nabla|^{s-s_0+}u_{\leq 1}\big\|_{L_t^\infty L_x^2} + \big\||\nabla|^{s-s_0+}u_{>1}\big\|_{L_t^\infty L_x^2} \\
&\lesssim \sum_{N \leq 1} \big\||\nabla|^{s-s_0+}u_N\big\|_{L_t^\infty L_x^2} + \sum_{N>1} \big\||\nabla|^{s-s_0+}u_N\big\|_{L_t^\infty L_x^2} \\
&\lesssim_u \sum_{N \leq 1} N^{s_0} N^{-s_0+} + \sum_{N>1} N^{(s-s_0+)-\frac{1}{2}} \lesssim_u 1.
\end{aligned}$$

To prove (6.5), by time-translation invariant, we only need to show that

$$\big\||\nabla|^s u_N(0)\big\|_{L_x^2} \lesssim_u N^{s_0} \quad \text{for all} \quad N > 0,\ s_0 := \frac{5}{r} - \frac{9}{2} > 0.$$

Using Duhamel formula (1.7) both forward and backward, we have

$$\begin{aligned}
\big\||\nabla|^s u_N(0)\big\|_{L_x^2} &= \Big\langle i\int_0^\infty e^{it\Delta}|\nabla|^s P_N F(u(t))\,\mathrm{d}t, -i\int_{-\infty}^0 e^{i\tau\Delta}|\nabla|^s P_N F(u(\tau))\,\mathrm{d}\tau\Big\rangle \\
&\leq \int_0^\infty \int_{-\infty}^0 \Big|\Big\langle e^{it\Delta}|\nabla|^s P_N F(u(t)),\ e^{i\tau\Delta}|\nabla|^s P_N F(u(\tau))\Big\rangle\Big|\,\mathrm{d}t\,\mathrm{d}\tau.
\end{aligned}$$

By Hölder's and the dispersive inequality

$$\begin{aligned}
&\Big|\Big\langle e^{it\Delta}|\nabla|^s P_N F(u(t)), e^{i\tau\Delta}|\nabla|^s P_N F(u(\tau))\Big\rangle\Big| \\
=\ &\Big|\Big\langle |\nabla|^s P_N F(u(t)), e^{i(\tau-t)\Delta}|\nabla|^s P_N F(u(\tau))\Big\rangle\Big| \\
\leq\ &\big\||\nabla|^s P_N F(u(t))\big\|_{L_x^r}\big\|e^{i(\tau-t)\Delta}|\nabla|^s P_N F(u(\tau))\big\|_{L_x^{r'}} \\
\lesssim\ &|\tau - t|^{5(\frac{1}{2}-\frac{1}{r})}\big\||\nabla|^s P_N F(u)\big\|_{L_x^r}^2.
\end{aligned}$$

On the other hand, from Bernstein's inequality

$$\begin{aligned}
&\Big|\Big\langle e^{it\Delta}|\nabla|^s P_N F(u(t)), e^{i\tau\Delta}|\nabla|^s P_N F(u(\tau))\Big\rangle\Big| \\
\leq\ &\big\||\nabla|^s P_N F(u)\big\|_{L_x^2}^2 \\
\lesssim\ &N^{10(\frac{1}{r}-\frac{1}{2})}\big\||\nabla|^s P_N F(u)\big\|_{L_x^r}^2.
\end{aligned}$$

Thus
$$\int_0^\infty \int_{-\infty}^0 \left|\left\langle e^{it\Delta}|\nabla|^s P_N F(u(t)), e^{i\tau\Delta}|\nabla|^s P_N F(u(\tau))\right\rangle\right| \mathrm{d}t\,\mathrm{d}\tau$$
$$\lesssim \big\||\nabla|^s F(u)\big\|^2_{L_t^\infty L_x^r} \int_0^\infty \int_{-\infty}^0 \min\{|\tau-t|^{5(\frac{1}{2}-\frac{1}{r})}, N^{10(\frac{1}{r}-\frac{1}{2})}\}\,\mathrm{d}t\,\mathrm{d}\tau$$
$$\lesssim \big\||\nabla|^s F(u)\big\|^2_{L_t^\infty L_x^r} N^{2(\frac{5}{r}-\frac{9}{2})} = \big\||\nabla|^s F(u)\big\|^2_{L_t^\infty L_x^r} N^{2s_0},$$
where we use the fact that $\frac{5}{2}-\frac{5}{r}<-2$.

With these propositions, we are now ready to complete the proof of Theorem 6.1. First, applying Proposition 6.2 with $s=\frac{1}{2}$, we obtain $u\in L_t^\infty \dot{H}_x^{\frac{1}{2}-s_0+}$ for some $s_0+>0$. By fractional chain rule and (6.1), we have $|\nabla|^{\frac{1}{2}-s_0+}F(u)\in L_t^\infty L_x^r$ for some $\frac{110}{101}\le r<\frac{10}{9}$. Again using Proposition 6.2 with $s=\frac{1}{2}-s_0+$, we have $u\in L_t^\infty \dot{H}_x^{\frac{1}{2}-2s_0+}$. By doing this with finite times, we will obtain $u\in L_t^\infty \dot{H}_x^{-\varepsilon}$ for some $0<\varepsilon<2s_0+$. This proves Theorem 6.1.

7 Low-to-high cascade

In this section we prove

Theorem 7.1 (Absence of cascade)**.** *There can not exist a global solution to* (1.1) *which is almost periodic modulo scaling, blows up both forward and backward and is low-to-high cascade in the sense of Theorem* 1.3*.*

Proof. We argue by contradiction. Assume there exists such an u. Then, by Theorem 6.1, $u\in L_t^\infty L_x^2$ and
$$0\le M(u)=M(u(t))=\int_{\mathbb{R}^5}|u(t,x)|^2\,\mathrm{d}x<\infty \quad \text{for all} \quad t\in\mathbb{R}.$$
Fix $t\in\mathbb{R}$. Let $\eta>0$ be sufficiently small. From (1.6)(Remark 1.1)
$$\int_{|\xi|\le c(\eta)N(t)}|\xi||\hat{u}(t,\xi)|^2\,\mathrm{d}\xi\le\eta.$$
Since $u\in L_t^\infty \dot{H}_x^{-\varepsilon}(\varepsilon>0)$, we see that
$$\int_{|\xi|\le c(\eta)N(t)}|\xi|^{-2\varepsilon}|\hat{u}(t,\xi)|^2\,\mathrm{d}\xi\lesssim 1.$$
Thus, by the interpolation theorem, we obtain
$$\int_{|\xi|\le c(\eta)N(t)}|\hat{u}(t,\xi)|^2\,\mathrm{d}\xi\lesssim_u \eta^{\frac{2\varepsilon}{1+2\varepsilon}}. \tag{7.1}$$
Meanwhile, it follows from the assumption (6.1) that
$$\int_{|\xi|\ge c(\eta)N(t)}|\hat{u}(t,\xi)|^2\,\mathrm{d}\xi\le[c(\eta)N(t)]^{-1}\int|\xi||\hat{u}(t,\xi)|^2\,\mathrm{d}\xi$$
$$\lesssim_u [c(\eta)N(t)]^{-1}.$$

This together with (7.1) and Plancherel's theorem yields

$$M(u) \lesssim [c(\eta)N(t)]^{-1} + \eta^{\frac{2\varepsilon}{1+2\varepsilon}} \quad \text{for all} \quad t \in \mathbb{R}.$$

As u is a low-to-high cascade solution, there exists $t_n \to \infty$ such that $N(t_n) \to \infty$. Since η is arbitrarily small, we conclude that $M(u) \equiv 0$. Thus, $u \equiv 0$, contradicting $\|u\|_{S(\mathbb{R})} = 0$.

8 Additional regularity for soliton

In order to preclude the final enemy, namely the soliton-like solution, we need to gain additional regularity to make the virial-type argument available.

Theorem 8.1. *Let u be a global radially symmetric solution to* (1.1) *that is almost periodic modulo scaling. Suppose also that $N(t) \equiv 1$ for all $t \in \mathbb{R}$. Then $u \in L_t^\infty \dot{H}_x^s$ for all $s \geq \frac{1}{2}$.*

To prove Theorem 8.1, we first develop some properties of the soliton-like solution.

Lemma 8.1 (Compactness in L_x^2)**.** *Let u be a soliton solution to* (1.1) *in the sense of Theorem* 1.3*. Then for any $\eta > 0$, there exists $C(\eta) > 0$ such that*

$$\sup_{t\in\mathbb{R}} \int_{|x|\geq C(\eta)} |u(t,x)|^2 \,\mathrm{d}x \leq \eta. \tag{8.1}$$

Proof. By negative regularity(Theorem 6.1),

$$\|u_{<N}(t)\|_{L_x^2(|x|\geq R)} \leq \|u_{<N}(t)\|_{L_x^2} \leq N^{\varepsilon}\big\||\nabla|^{-\varepsilon}u\big\|_{L_t^\infty L_x^2} \lesssim_u N^{\varepsilon}.$$

This can be made smaller than η by choosing $N = N(\eta)$ sufficiently small.

To estimate the contribution of high frequency, using Schur's test lemma

$$\big\|\chi_{|x|\geq 2R}(-\Delta)^{-\frac{1}{2}}|\nabla|^{\frac{1}{2}}P_{\geq N}\chi_{|x|\leq R}\big\|_{L^2\to L^2} \lesssim N^{-\frac{1}{2}}\langle RN\rangle^{-m}.$$

On the other hand, by Bernstein's inequality

$$\big\|\chi_{|x|\geq 2R}(-\Delta)^{-\frac{1}{2}}|\nabla|^{\frac{1}{2}}P_{\geq N}\chi_{|x|\geq R}\big\|_{L^2\to L^2} \lesssim N^{-\frac{1}{2}}.$$

Thus,

$$\begin{aligned}
&\int_{|x|\geq 2R} |u_{\geq N}(t,x)|^2 \,\mathrm{d}x \\
\lesssim &\int_{|x|\geq 2R} \big|(-\Delta)^{-\frac{1}{2}}|\nabla|^{\frac{1}{2}}P_{\geq N}\chi_{\leq R}|\nabla|^{\frac{1}{2}}u_{\geq N}\big|^2 \,\mathrm{d}x \\
&\quad + \int_{|x|\geq 2R} \big|(-\Delta)^{-\frac{1}{2}}|\nabla|^{\frac{1}{2}}P_{\geq N}\chi_{\geq R}|\nabla|^{\frac{1}{2}}u_{\geq N}\big|^2 \,\mathrm{d}x \\
\lesssim_u &\; N^{-1}\langle RN\rangle^{-2m} + N^{-1}\int_{|x|\geq 2R} \big||\nabla|^{\frac{1}{2}}u\big|^2 \,\mathrm{d}x.
\end{aligned}$$

Choosing R sufficiently large, the first term on the right hand side can be made smaller than η. By Definition 1.2, the second term can also be smaller that η. Thus, it concludes (8.1).

Lemma 8.2 (Spacetime bounds)**.** *Let $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ be a maximal life-span solution to* (1.1) *which is almost periodic modulo scaling. Let J be any subinterval of I. Then for any L^2-admissible pair (q, r)*

$$\int_J N(t)^2 \, \mathrm{d}t \lesssim \int_J \Big(\int_{\mathbb{R}^5} \big| |\nabla|^{\frac{1}{2}} u(t,x) \big|^r \, \mathrm{d}x \Big)^{q/r} \mathrm{d}t \lesssim 1 + \int_J N(t)^2 \, \mathrm{d}t \tag{8.2}$$

Proof. As noted, the proof can be found in [13], [14]. For the sake of convenience, we give a self-contained argument using the ideas in them.

We first prove the second inequality. Let $\eta > 0$ be chosen later, divide J into subintervals I_j such that on each I_j

$$\int_{I_j} N(t)^2 \, \mathrm{d}t \le \eta.$$

By pigeonhole principle, there are at most $m \le \eta^{-1} \times \big(1 + \int_J N(t)^2 \, \mathrm{d}t\big)$ subintervals. For each j, choose t_j such that

$$N(t_j)^2 |I_j| \le 2\eta. \tag{8.3}$$

By Strichartz's estimate, the Hardy-Littlewood-Sobolev, and Hölder's, Sobolev's inequality, we have on $I_j \times \mathbb{R}^5$ that

$$\begin{aligned}
\big\| |\nabla|^{\frac{1}{2}} u \big\|_{L_t^q L_x^r} \le \ & \big\| e^{i(t-t_j)\Delta} |\nabla|^{\frac{1}{2}} u(t_j) \big\|_{L_t^q L_x^r} \\
& + \Big\| \int_{t_j}^t e^{i(t-\tau)\Delta} |\nabla|^{\frac{1}{2}} F(u(\tau)) \, \mathrm{d}\tau \Big\|_{L_t^q L_x^r} \\
\lesssim \ & \big\| |\nabla|^{\frac{1}{2}} u_{\ge N_0}(t_j) \big\|_2 + \big\| e^{i(t-t_j)\Delta} |\nabla|^{\frac{1}{2}} u_{\le N_0}(t_j) \big\|_{L_t^q L_x^r} \\
& + \big\| |\nabla|^{\frac{1}{2}} F(u) \big\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \\
\lesssim \ & \big\| |\nabla|^{\frac{1}{2}} u_{\ge N_0}(t_j) \big\|_2 + |I_j|^{1/q} N_0^{2/q} \big\| |\nabla|^{\frac{1}{2}} u_{< N_0} \big\|_{L_t^\infty L_x^2} \\
& + \big\| |\nabla|^{\frac{1}{2}} u \big\|_{L_t^q L_x^r}^3,
\end{aligned}$$

where $\tilde{q}' = q/3$, $\tilde{r}' = (15 - 3r)/5r$. From the definition of almost periodic modulo scaling, choosing N_0 as a large multiple of $N(t_j)$, then the first term on the right hand side can be made as small as we wish. Invoking (8.3) and choosing η sufficiently small, the second term can also be made sufficiently small. Thus, by bootstrap argument, we obtain

$$\int_{I_j} \Big(\int_{\mathbb{R}^5} \big| |\nabla|^{\frac{1}{2}} u(t,x) \big|^r \, \mathrm{d}x \Big)^{q/r} \mathrm{d}t \le \eta.$$

Recalling the bound on subinterval number, we have

$$\int_J \Big(\int_{\mathbb{R}^5} \big| |\nabla|^{\frac{1}{2}} u(t,x) \big|^r \, \mathrm{d}x \Big)^{q/r} \mathrm{d}t \le 1 + \int_J N(t)^2 \, \mathrm{d}t.$$

For the first inequality, note that by Definition 1.2, we must have

$$\int_{|x| \le C(\eta) N(t)^{-1}} \big| |\nabla|^{\frac{1}{2}} u(t,x) \big|^2 \, \mathrm{d}x \gtrsim_u 1.$$

Using Hölder's inequality

$$\Big(\int_{\mathbb{R}^5}\big||\nabla|^{\frac12}u(t,x)\big|^r\,\mathrm{d}x\Big)^{1/r}\gtrsim\Big(\int_{|x|\le C(\eta)N(t)^{-1}}\big||\nabla|^{\frac12}u(t,x)\big|^2\,\mathrm{d}x\Big)^{1/2}N(t)^{2/q}\gtrsim_u N(t)^{2/q}.$$

Integrating the above inequality on J, we have

$$\int_J\Big(\int_{\mathbb{R}^5}\big||\nabla|^{\frac12}u(t,x)\big|^r\,\mathrm{d}x\Big)^{q/r}\,\mathrm{d}t\gtrsim_u\int_J N(t)^2\,\mathrm{d}t.$$

Remark 8.1. *We have for all $\dot H^{1/2}$-admissible pairs $(q,\,r)$ that*

$$\int_J N(t)^2\,\mathrm{d}t\lesssim\|u\|^q_{L^q_tL^r_x(J\times\mathbb{R}^5)}\lesssim 1+\int_J N(t)^2\,\mathrm{d}t.$$

Indeed,

$$\begin{aligned}\|u\|_{L^q_tL^r_x}\le\ &\big\|e^{i(t-t_j)\Delta}|\nabla|^{\frac12}u(t_j)\big\|_{L^q_tL^r_x}\\&+\Big\|\int_{t_j}^t e^{i(t-\tau)\Delta}|\nabla|^{\frac12}F(u(\tau))\,\mathrm{d}\tau\Big\|_{L^q_tL^r_x}\\\lesssim\ &\big\||\nabla|^{\frac12}u_{\ge N_0}(t_j)\big\|_2+|I_j|^{1/q}N_0^{2/q}\big\||\nabla|^{\frac12}u_{<N_0}\big\|_{L^\infty_tL^2_x}\\&+\|u\|^2_{L^q_tL^r_x}\big\||\nabla|^{\frac12}u\big\|_{L^{q_1}_tL^{r_1}_x},\end{aligned}$$

where $(q_1,\,r_1)$ is an L^2-admissible pair. Using the same argument as that in proving (8.2), we easily get the bounds.

Due to this proposition, we could obtain some local estimates for the soliton-like solution. Specifically, we have for L^2-admissible pair (q,r) and $\dot H^{1/2}$-admissible pair $(\tilde q,\tilde r)$ that

$$\|u\|_{L^{\tilde q}_tL^{\tilde r}_x(J\times\mathbb{R}^5)}\lesssim_u\langle|J|\rangle^{\frac{1}{\tilde q}},\quad\big\||\nabla|^{\frac12}u\big\|_{L^q_tL^r_x(J\times\mathbb{R}^5)}\lesssim_u\langle|J|\rangle^{\frac1q}.\tag{8.4}$$

By the Hardy-Littlewood-Sobolev inequality and the interpolation

$$\begin{aligned}\|F(u)\|_{L^2_tL^{10/7}_x}&\le\|(|\cdot|^{-3}*|u|^2)\|_{L^2_tL^{10/3}_x}\|u\|_{L^\infty_tL^{5/2}_x}\lesssim_u\|u\|^2_{L^4_tL^{20/7}_x}\\&\lesssim_u\|u\|_{L^4_tL^{10/3}_x}\big\||\nabla|^{\frac12}u\big\|_{L^4_tL^{5/2}_x}\lesssim_u\langle|J|\rangle^{\frac12}.\end{aligned}\tag{8.5}$$

By the weighted Strichartz estimate

$$\big\||x|^2u\big\|_{L^4_tL^\infty_x}\lesssim_u\langle|J|\rangle^{\frac12}.\tag{8.6}$$

From Definition 1.2

$$\lim_{N\to\infty}\|u_{\ge N}\|_{L^\infty_t\dot H^{1/2}_x(\mathbb{R}\times\mathbb{R}^5)}=0.\tag{8.7}$$

Now, define

$$G(N):=\|u_{\ge N}\|_{L^\infty_t\dot H^{1/2}_x(\mathbb{R}\times\mathbb{R}^5)}\tag{8.8}$$

Note that

$$\lim_{N\to\infty}G(N)=0.\tag{8.9}$$

To prove Theorem 8.1, it suffices to prove that $G(N) \lesssim_u N^{-s}$ holds for any $s > 0$ and any sufficiently large N, since we consequently have $\|u_N\|_{L_t^\infty \dot{H}_x^{s+1/2}} \lesssim N^s \|u_N\|_{L_t^\infty \dot{H}_x^{1/2}} \lesssim_u 1$. This will be achieved by iterating the following proposition with sufficiently small η.

Proposition 8.1. *Let u be as in Theorem 8.1. Let $\eta > 0$ be sufficiently small. Then, for sufficiently large $N = N(\eta, u)$, we have*

$$G(N) \lesssim_u \eta G(\frac{N}{16}). \tag{8.10}$$

To prove the proposition, it suffices to prove

$$\|u_{\geq N}(t)\|_{\dot{H}_x^{1/2}} \lesssim_u \eta G\left(\frac{N}{16}\right) \tag{8.11}$$

for all $t \in \mathbb{R}$ and all N sufficiently large. By time-translation invariant, we may set $t = 0$. Using Duhamel formula (1.7) and the in/out decomposition

$$\begin{aligned} |\nabla|^{\frac{1}{2}} u_{\geq N}(0) &= (P^+ + P^-)|\nabla|^{\frac{1}{2}} u_{\geq N}(0) \\ &= \lim_{T\to\infty} i \int_0^T P^+ e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\,\mathrm{d}t \\ &\quad - \lim_{T\to\infty} \int_{-T}^0 P^- e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\,\mathrm{d}t \end{aligned} \tag{8.12}$$

as weak limits in L_x^2. Using the property of weak closedness for unit ball, namely

$$f_T \rightharpoonup f \quad \Longrightarrow \quad \|f\| \leq \liminf_T \|f_T\|,$$

we are reduced to proving that RHS of (8.12) $\lesssim_u$ RHS of (8.11).

Note that $P^{\pm}$ are singular at $x = 0$; to get around this, we introduce the cutoff $\psi_N(x) := \psi(N|x|)$, where ψ is the characteristic function of $[1, \infty)$. As the short times and large times will be treated differently, we rewrite (8.12) as

$$\begin{aligned} |\nabla|^{\frac{1}{2}} u_{\geq N}(0) &= [\psi_N(x) + (1 - \psi_N(x))]|\nabla|^{\frac{1}{2}} u_{\geq N}(0) \\ &= \lim_{T\to\infty} \int_0^T \psi_N(x) P^+ e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\,\mathrm{d}t \\ &\quad - \lim_{T\to\infty} i \int_{-T}^0 \psi_N(x) P^- e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\,\mathrm{d}t \\ &\quad + \lim_{T\to\infty} i \int_0^T (1 - \psi_N(x)) e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\,\mathrm{d}t, \end{aligned}$$

$$
\begin{aligned}
|\nabla|^{\frac{1}{2}} u_{\geq N}(0) \quad = \quad & i \int_0^\delta \psi_N(x) P^+ e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\, \mathrm{d}t \\
& -i \int_{-\delta}^0 \psi_N(x) P^- e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\, \mathrm{d}t \\
& +i \int_0^\delta (1-\psi_N(x)) e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\, \mathrm{d}t \\
& + \lim_{T\to\infty} \sum_{M \geq N} i \int_\delta^T \int_{\mathbb{R}^5} \psi_N [P_M^+ e^{-it\Delta}](x,y) \tilde{P}_M |\nabla|^{\frac{1}{2}} F(u(t))\, \mathrm{d}y\, \mathrm{d}t \\
& - \lim_{T\to\infty} \sum_{M \geq N} i \int_{-T}^{-\delta} \int_{\mathbb{R}^5} \psi_N [P_M^- e^{-it\Delta}](x,y) \tilde{P}_M |\nabla|^{\frac{1}{2}} F(u(t))\, \mathrm{d}y\, \mathrm{d}t \\
& + \lim_{T\to\infty} \sum_{M \geq N} i \int_\delta^T \int_{\mathbb{R}^5} (1-\psi_N) [\tilde{P}_M e^{-it\Delta}](x,y) P_M |\nabla|^{\frac{1}{2}} F(u(t))\, \mathrm{d}y\, \mathrm{d}t \\
:= \quad & I_1 - I_2 + I_3 + I_4 - I_5 + I_6. \qquad (8.13)
\end{aligned}
$$

Note that we used the identity

$$
P_{\geq N} = \sum_{M \geq N} P_M \tilde{P}_M,
$$

where $\tilde{P}_M := P_{M/2} + P_M + P_{2M}$.

For integrals over short times, namely I_1, I_2, I_3, we have the following estimate, that is

Lemma 8.3 (Local estimate)**.** *For any sufficiently small $\eta > 0$, there exists $\delta = \delta(u, \eta) > 0$ such that*

$$
\Big\| \int_0^\delta e^{-it\Delta} P_{\geq N} |\nabla|^{\frac{1}{2}} F(u(t))\, \mathrm{d}t \Big\|_{L_x^2} \lesssim_u \eta G\left(\frac{N}{8}\right)
$$

for sufficiently large N depending on u and η. An analogous estimate holds for integration over $[-\delta, 0]$ and after pre-multiplication by $P^{\pm}$.

Proof. By Strichartz's estimate, it only needs to prove

$$
\big\| |\nabla|^{\frac{1}{2}} P_{\geq N} F(u) \big\|_{L_t^2 L_x^{10/7}(J \times \mathbb{R}^5)} \lesssim_u \eta G\left(\frac{N}{8}\right)
$$

for any time interval J with $|J| \leq \delta$.

From (8.9), for any $\eta > 0$, there exists $N_0 = N_0(u, \eta)$ such that

$$
\|u_{\geq N_0}\|_{L_t^\infty \dot{H}_x^{1/2}} \leq \eta. \qquad (8.14)
$$

Let $N \geq 8N_0$. Decompose u as

$$
u := u_{\geq \frac{N}{8}} + u_{N_0 \leq \cdot < \frac{N}{8}} + u_{< N_0},
$$

and make a corresponding expansion of $P_{\geq N}F(u)$. Note that any term in the resulting expansion does not contain $u_{\geq \frac{N}{8}}$ vanishes.

We first consider a term with two factors of the form $u_{<N_0}$. Using Hölder's inequality, the fractional Leibniz rule, the Hardy-Littlewood-Sobolev, and Bernstein's inequality

$$\begin{aligned}
& \big\||\nabla|^{\frac{1}{2}}(|\nabla|^{-2}(u_{<N_0}^2)u_{\geq \frac{N}{8}})\big\|_{L_t^2L_x^{10/7}(J\times\mathbb{R}^5)} \\
\leq\ & \big\||\nabla|^{-2}(u_{<N_0}^2)\big\|_{L_t^2L_x^5(J\times\mathbb{R}^5)}\big\||\nabla|^{\frac{1}{2}}u_{\geq \frac{N}{8}}\big\|_{L_t^\infty L_x^2} \\
& + \big\||\nabla|^{-\frac{3}{2}}(u_{<N_0}^2)\big\|_{L_t^2L_x^{10/3}(J\times\mathbb{R}^5)}\|u_{\geq \frac{N}{8}}\|_{L_t^\infty L_x^{5/2}} \\
\lesssim\ & \|u_{<N_0}^2\|_{L_t^2L_x^{5/3}(J\times\mathbb{R}^5)}G(\frac{N}{8}) + \|u_{<N_0}^2\|_{L_t^2L_x^{5/3}(J\times\mathbb{R}^5)}G(\frac{N}{8}) \\
\lesssim_u\ & |J|^{\frac{1}{2}}N_0G(\frac{N}{8}),
\end{aligned}$$

and

$$\begin{aligned}
& \big\||\nabla|^{\frac{1}{2}}(|\nabla|^{-2}(u_{<N_0}u_{\geq \frac{N}{8}})u_{<N_0})\big\|_{L_t^2L_x^{10/7}(J\times\mathbb{R}^5)} \\
\leq\ & \big\||\nabla|^{-2}(u_{<N_0}u_{\geq \frac{N}{8}})\big\|_{L_t^4L_x^{10/3}}\big\||\nabla|^{\frac{1}{2}}u_{<N_0}\big\|_{L_t^4L_x^{5/2}} \\
& + \big\||\nabla|^{-\frac{3}{2}}(u_{<N_0}u_{\geq \frac{N}{8}})\big\|_{L_t^4L_x^{5/2}}\|u_{<N_0}\|_{L_t^4L_x^{10/3}} \\
\lesssim_u\ & \|u_{<N_0}u_{\geq \frac{N}{8}}\|_{L_t^4L_x^{10/7}}|J|^{\frac{1}{4}}N_0^{\frac{1}{2}} + \|u_{<N_0}u_{\geq \frac{N}{8}}\|_{L_t^4L_x^{10/7}}|J|^{\frac{1}{4}}N_0^{\frac{1}{2}} \\
\lesssim_u\ & \|u_{<N_0}\|_{L_t^4L_x^{10/3}}\|u_{\geq \frac{N}{8}}\|_{L_t^\infty L_x^{5/2}}|J|^{\frac{1}{4}}N_0^{\frac{1}{2}} \\
\lesssim_u\ & |J|^{\frac{1}{2}}N_0G(\frac{N}{8}).
\end{aligned}$$

Choosing δ sufficiently small depending on η and N_0, we see they are acceptable.

Now, we have to estimate those components of $P_{\geq N}F(u)$ which involve $u_{\geq \frac{N}{8}}$ and at least one of the other terms is not $u_{<N_0}$. Using Hölder's inequality, the fractional Leibniz rule, the Hardy-Littlewood-Sobolev, Bernstein's inequality, (8.4), (8.14),

$$\begin{aligned}
& \big\||\nabla|^{\frac{1}{2}}(|\nabla|^{-2}(u_{\geq N_0}u_{\geq \frac{N}{8}})u)\big\|_{L_t^2L_x^{10/7}(J\times\mathbb{R}^5)} \\
\lesssim\ & \big\||\nabla|^{-\frac{3}{2}}(u_{\geq N_0}u_{\geq \frac{N}{8}})\big\|_{L_t^4L_x^{5/2}(J\times\mathbb{R}^5)}\|u\|_{L_t^4L_x^{10/3}(J\times\mathbb{R}^5)} \\
& + \big\||\nabla|^{-2}(u_{\geq N_0}u_{\geq \frac{N}{8}})\big\|_{L_t^\infty L_x^5(J\times\mathbb{R}^5)}\big\||\nabla|^{\frac{1}{2}}u\big\|_{L_t^2L_x^2} \\
\lesssim_u\ & \|u_{\geq N_0}u_{\geq \frac{N}{8}}\|_{L_t^4L_x^{10/7}(J\times\mathbb{R}^5)}\langle|J|\rangle^{\frac{1}{4}} + |J|^{\frac{1}{2}}\|u_{\geq N_0}u_{\geq \frac{N}{8}}\|_{L_t^\infty L_x^{5/3}(J\times\mathbb{R}^5)} \\
\lesssim_u\ & \|u_{\geq \frac{N}{8}}\|_{L_t^\infty L_x^{5/2}}\|u_{\geq N_0}\|_{L_t^4L_x^{10/3}(J\times\mathbb{R}^5)}\langle|J|\rangle^{\frac{1}{4}} + |J|^{\frac{1}{2}}\|u_{\geq \frac{N}{8}}\|_{L_t^\infty L_x^{5/2}}\|u_{\geq N_0}\|_{L_t^\infty L_x^5(J\times\mathbb{R}^5)} \\
\lesssim_u\ & \langle|J|\rangle^{\frac{1}{4}}\|u_{\geq N_0}\|_{L_t^4L_x^{10/3}(J\times\mathbb{R}^5)}G(\frac{N}{8}) + \eta|J|^{\frac{1}{2}}N_0G(\frac{N}{8})
\end{aligned}$$

By (8.4)

$$\|u_{\geq N_0}\|_{L_t^2L_x^5(J\times\mathbb{R}^5)} \lesssim \langle|J|\rangle^{\frac{1}{2}}.$$

Hence, interpolating with (8.14), we have

$$\|u_{\geq N_0}\|_{L_t^4 L_x^{10/3}(J\times\mathbb{R}^5)} \lesssim \eta^{\frac{1}{2}}\langle |J|\rangle^{\frac{1}{4}}.$$

Thus, we obtain

$$\big\||\nabla|^{\frac{1}{2}}(|\nabla|^{-2}(u_{\geq N_0}u_{\geq\frac{N}{8}})u)\big\|_{L_t^2 L_x^{10/7}(J\times\mathbb{R}^5)} \lesssim_u \eta^{\frac{1}{2}}\langle |J|\rangle^{\frac{1}{2}}G\big(\frac{N}{8}\big)+\eta|J|^{\frac{1}{2}}N_0 G\big(\frac{N}{8}\big),$$

which is acceptable.

In the same manner, we estimate

$$\begin{aligned}
&\big\||\nabla|^{\frac{1}{2}}(|\nabla|^{-2}(u_{\geq\frac{N}{8}}u)u_{\geq N_0})\big\|_{L_t^2 L_x^{10/7}(J\times\mathbb{R}^5)}\\
&\lesssim \big\||\nabla|^{-\frac{3}{2}}(u_{\geq\frac{N}{8}}u)\big\|_{L_t^2 L_x^{10/3}(J\times\mathbb{R}^5)}\|u_{\geq N_0}\|_{L_t^\infty L_x^{5/2}}\\
&\qquad+\big\||\nabla|^{-2}(u_{\geq\frac{N}{8}}u)\big\|_{L_t^\infty L_x^5(J\times\mathbb{R}^5)}\big\||\nabla|^{\frac{1}{2}}u_{\geq N_0}\big\|_{L_t^2L_x^2}\\
&\lesssim_u \eta\|u\|_{L_t^2L_x^5(J\times\mathbb{R}^5)}\|u_{\geq\frac{N}{8}}\|_{L_t^\infty L_x^{5/2}} \lesssim_u \eta\langle |J|\rangle^{\frac{1}{2}}G\big(\frac{N}{8}\big).
\end{aligned}$$

Another term $\big\||\nabla|^{\frac{1}{2}}\big(|\nabla|^{-2}(u_{\geq N_0}u)u_{\geq\frac{N}{8}}\big)\big\|_{L_t^2L_x^{10/7}(J\times\mathbb{R}^5)}$ can be estimated similarly. This concludes the proof of Lemma 8.3.

We now turn our attention to I_4, I_5, I_6, namely the integrations over large times: $|t|\geq\delta$. Making use of the properties of the kernels $P_Me^{-it\Delta}$, $P_M^{\pm}e^{-it\Delta}$(see Lemma 2.4, Lemma 2.7), we break the regions of (t,y) integration into two pieces: $|y|\gtrsim M|t|$ and $|y|\ll M|t|$. when $|x|\geq N^{-1}$, we use the kernel $P_M^{\pm}e^{-it\Delta}$; in this case $|y|-|x|\sim M|t|$ implies $|y|\gtrsim M|t|$ for $|t|\geq\delta\geq N^{-2}$. When $|x|\leq N^{-1}$, we use $P_Me^{-it\Delta}$; in this case $|y-x|\sim M|t|$ implies $|y|\gtrsim M|t|$ for $|t|\geq\delta\geq N^{-2}$. The condition $\delta\geq N^{-2}$ can be satisfied under our statement N sufficiently large depending on u and η.

Define χ_k as the characteristic function of the set

$$\{\,(t,y): 2^k\delta\leq|t|\leq 2^{k+1}\delta, |y|\gtrsim M|t|\,\}.$$

Then we have the following estimate

Lemma 8.4 (Main contribution)**.** *Let $\eta>0$ be a small number and δ be as in Lemma 8.3. Then*

$$\sum_{M\geq N}\sum_{k=0}^{\infty}\Big\|\int_{2^k\delta}^{2^{k+1}\delta}\int_{\mathbb{R}^5}[P_Me^{-it\Delta}](x,y)\chi_k(t,y)[\tilde{P}_M|\nabla|^{\frac{1}{2}}F(u(t))](y)\,\mathrm{d}y\,\mathrm{d}t\Big\|_{L_x^2}\lesssim_u \eta G\big(\frac{N}{16}\big) \tag{8.15}$$

for all N sufficiently large depending on u and η. An analogous estimate holds for integration over $[-2^{k+1}\delta,-2^k\delta]$ and with P_M replaced by $P_M^{\pm}$.

Proof. By Strichartz's estimates

$$\begin{aligned}
&\Big\|\int_{2^k\delta}^{2^{k+1}\delta}\int_{\mathbb{R}^5}[P_Me^{-it\Delta}](x,y)\chi_k(t,y)[\tilde{P}_M|\nabla|^{\frac{1}{2}}F(u(t))](y)\,\mathrm{d}y\,\mathrm{d}t\Big\|_{L_x^2}\\
&\lesssim \big\|\chi_k\tilde{P}_M(|\nabla|^{\frac{1}{2}}F(u))\big\|_{L_t^2L_y^{10/7}([2^k\delta,\,2^{k+1}\delta]\times\mathbb{R}^5)}
\end{aligned}$$

Using the fractional Leibniz rule, we turn to estimate

$$\mathrm{II}_1 = \big\|\chi_k \tilde{P}_M(|\nabla|^{-\frac{3}{2}}(|u|^2)u)\big\|_{L_t^2 L_x^{10/7}([2^k\delta,\, 2^{k+1}\delta]\times\mathbb{R}^5)},$$

$$\mathrm{II}_2 = \big\|\chi_k \tilde{P}_M(|\nabla|^{-2}(|u|^2)|\nabla|^{\frac{1}{2}}u)\big\|_{L_t^2 L_x^{10/7}([2^k\delta,\, 2^{k+1}\delta]\times\mathbb{R}^5)}.$$

We only estimate II_1, since II_2 can be treated similarly, using the fact that $u \in L_t^\infty H^{1/2}$.

Write u as $u := u_{\leq \frac{M}{16}} + u_{> \frac{M}{16}}$. In what follows, all spacetime norms are taken on the slab $[2^k\delta,\ 2^{k+1}\delta] \times \mathbb{R}^5$, unless noted otherwise. Using the support property of $\tilde{P}_M$, II_{11} can be controlled by

$$\begin{aligned}
\mathrm{II}_1 &\lesssim \big\|\chi_k|\nabla|^{-\frac{3}{2}}(u^2)u_{>\frac{M}{16}}\big\|_{L_t^2L_x^{10/7}} + \big\|\chi_k|\nabla|^{-\frac{3}{2}}(uu_{>\frac{M}{16}})u_{\leq\frac{M}{16}}\big\|_{L_t^2L_x^{10/7}} \\
&:= \mathrm{II}_{11} + \mathrm{II}_{12}.
\end{aligned}$$

Using Hölder's inequality, and (8.6), we have

$$\begin{aligned}
&\big\|\chi_k|\nabla|^{-\frac{3}{2}}(u^2)u_{>\frac{M}{16}}\big\|_{L_t^2L_x^{10/7}} \leq \big\|u_{>\frac{M}{16}}\big\|_{L_t^\infty L_x^{5/2}}\big\|\chi_k|\nabla|^{-\frac{3}{2}}(u^2)\big\|_{L_t^2L_x^{10/3}} \\
\lesssim\ & G(\frac{M}{16})\left(\Big\|\chi_k\int_{|x-y|\geq\frac{|y|}{2}}\frac{|u(x)|^2}{|x-y|^{7/2}}\,\mathrm{d}x\Big\|_{L_t^2L_y^{10/3}} + \Big\|\chi_k\int_{|x-y|<\frac{|y|}{2}}\frac{|u(x)|^2}{|x-y|^{7/2}}\,\mathrm{d}x\Big\|_{L_t^2L_y^{10/3}}\right) \\
\lesssim\ & G(\frac{M}{16})\left(\|\chi_k|y|^{-\frac{7}{2}}\|_{L_t^2L_y^{10/3}}\|u\|_{L_t^\infty L_x^2} + \Big\|\chi_k|y|^{-\frac{16}{5}}\int_{|x-y|<\frac{|y|}{2}}\frac{|y|^{16/5}|u|^2}{|x-y|^{7/2}}\,\mathrm{d}x\Big\|_{L_t^2L_y^{10/3}}\right) \\
\lesssim_u\ & G(\frac{M}{16})\left(M^{-2}(2^k\delta)^{-\frac{3}{2}} + \Big\|\chi_k|y|^{-\frac{16}{5}}\big\|1_{\leq\frac{|y|}{2}}|\cdot|^{-\frac{7}{2}}\big\|_{L_x^{5/4}}\big\||y|^2u\big\|_{L_x^\infty}^{\frac{8}{5}}\|u\|_{L_x^2}^{\frac{2}{5}}\Big\|_{L_t^2L_y^{10/3}}\right) \\
\lesssim_u\ & G(\frac{M}{16})\left(M^{-2}(2^k\delta)^{-\frac{3}{2}} + \big\|\chi_k|y|^{-\frac{27}{10}}\big\|_{L_t^{10}L_y^{10/3}}\big\||y|^2u\big\|_{L_t^4L_x^\infty}^{\frac{8}{5}}\right) \\
\lesssim_u\ & G(\frac{M}{16})\left(M^{-2}(2^k\delta)^{-\frac{3}{2}} + M^{-\frac{6}{5}}(2^k\delta)^{-\frac{11}{10}}\langle 2^k\delta\rangle^{\frac{4}{5}}\right).
\end{aligned}$$

Using the Hardy-Littlewood-Sobolev, Hölder's inequality, (8.6), we estimate II_{12} as the following :

$$\begin{aligned}
\mathrm{II}_{12} &\leq \|\chi_k u\|_{L_t^2L_x^5}\big\||\nabla|^{-\frac{3}{2}}(uu_{>\frac{M}{16}})\big\|_{L_t^\infty L_x^2} \\
&\lesssim \|\chi_k|y|^{-2}\|_{L_t^4L_x^5}\big\||y|^2u\big\|_{L_t^4L_x^\infty}\|uu_{>\frac{M}{16}}\|_{L_t^\infty L_x^{5/4}} \\
&\lesssim_u M^{-1}(2^k\delta)^{-\frac{3}{4}}\langle 2^k\delta\rangle^{\frac{1}{2}}G(\frac{M}{16}).
\end{aligned}$$

Thus, the left hand side of (8.15) can be bounded by:

$$\big(N^{-\frac{6}{5}}\delta^{-\frac{11}{10}} + N^{-\frac{6}{5}}\delta^{-\frac{3}{10}} + N^{-2}\delta^{-\frac{3}{2}} + N^{-1}\delta^{-\frac{3}{4}} + N^{-1}\delta^{-\frac{1}{2}}\big)G(\frac{N}{16}).$$

This is acceptable by choosing N sufficiently large depending on δ and η.

The last claim follows from the time reversal symmetry and the L_x^2-boundedness of $P^{\pm}$.

We now turn to the region of (t,y) integration where $|y| \ll M|t|$. To begin with, we recall the bounds in [15] for the kernels of the propagators in the region $|x| \leq N^{-1}$, $|y| \ll M|t|$, $|t| \geq \delta \gg N^{-2}$; and the region $|x| \geq N^{-1}$, y and t as above:

$$\left|P_M e^{-it\Delta}(x,y)\right| + \left|P_M^{\pm} e^{-it\Delta}(x,y)\right| \lesssim \frac{1}{(M^2|t|)^{50}} K_M(x,y),$$

where

$$K_M(x,y) := \frac{M^5}{\langle M(x-y)\rangle^{50}} + \frac{M^5}{\langle Mx\rangle^2 \langle My\rangle^2 \langle M|x| - M|y|\rangle^{50}}$$

be bounded on L_x^2.

Let $\tilde{\chi}_k$ be the characteristic function of the set

$$\{\,(t,y) : 2^k\delta \leq |t| \leq 2^{k+1}\delta,\ |y| \ll M|t|\,\}.$$

Lemma 8.5 (The tail)**.** *Let $\eta > 0$ be a small number and δ be as in Lemma 8.3. Then*

$$\sum_{M \geq N} \sum_{k=0}^{\infty} \Big\| \int_{2^k\delta}^{2^{k+1}\delta} \int_{\mathbb{R}^5} \frac{K_M(x,y)}{(M^2|t|)^{50}} \tilde{\chi}_k(t,y) [\tilde{P}_M |\nabla|^{\frac{1}{2}} F(u(t))](y)\, \mathrm{d}y\, \mathrm{d}t \Big\|_{L_x^2} \lesssim_u \eta G\big(\frac{N}{16}\big)$$

for sufficiently large N depending on u and η.

Proof. By Minkowski's inequality, the boundedness of K_M, the support property of $\tilde{P}_M$, Hölder's and the Hardy-Littlewood-Sobolev inequality

$$\begin{aligned}
&\Big\| \int_{2^k\delta}^{2^{k+1}\delta} \int_{\mathbb{R}^5} \frac{K_M(x,y)}{(M^2|t|)^{50}} \tilde{\chi}_k(t,y) [\tilde{P}_M |\nabla|^{\frac{1}{2}} F(u(t))](y)\, \mathrm{d}y\, \mathrm{d}t \Big\|_{L_x^2} \\
\lesssim\ & (M^2 2^k\delta)^{-50} \big\| \tilde{\chi}_k(t,y) [\tilde{P}_M |\nabla|^{\frac{1}{2}} F(u)] \big\|_{L_t^1 L_y^2} \\
\lesssim\ & (M^2 2^k\delta)^{-50} 2^k\delta M^{\frac{1}{2}} \Big\| \tilde{P}_M \big(|\nabla|^{-2} \big(\big| u_{\leq \frac{M}{16}} + u_{> \frac{M}{16}} \big|^2 \big) (u_{\leq \frac{M}{16}} + u_{> \frac{M}{16}}) \big) \Big\|_{L_t^\infty L_y^2} \\
\lesssim\ & (M^2 2^k\delta)^{-50} 2^k\delta M^{\frac{1}{2}} \Big(\big\| |\nabla|^{-2}(u^2) u_{> \frac{M}{16}} \big\|_{L_t^\infty L_y^2} + \big\| |\nabla|^{-2}(u u_{> \frac{M}{16}}) u_{\leq \frac{M}{16}} \big\|_{L_t^\infty L_y^2} \Big) \\
\lesssim\ & (M^2 2^k\delta)^{-50} 2^k\delta M^{\frac{1}{2}} \Big(\big\| |\nabla|^{-2}(u^2) \big\|_{L_t^\infty L_x^{5/2}} \| u_{> \frac{M}{16}} \|_{L_t^\infty L_x^{10}} \\
&\qquad\qquad + \big\| |\nabla|^{-2}(u u_{> \frac{M}{16}}) \big\|_{L_t^\infty L_x^{10}} \|u\|_{L_t^\infty L_x^{5/2}} \Big) \\
\lesssim_u\ & (M^2 2^k\delta)^{-50} 2^k\delta M^{\frac{1}{2}} \Big(\|u\|_{L_t^\infty L_x^{5/2}}^2 M^{\frac{3}{2}} G\big(\frac{M}{16}\big) + \|u_{> \frac{M}{16}}\|_{L_t^\infty L_x^{10}} \|u\|_{L_t^\infty L_x^{5/2}} \Big) \\
\lesssim_u\ & (M^2 2^k\delta)^{-50} 2^k\delta M^2 G\big(\frac{M}{16}\big)
\end{aligned}$$

Summing first over $k \geq 0$ and then $M \geq N$, we obtain

$$\begin{aligned}
\sum_{M \geq N} \sum_{k=0}^{\infty} \Big\| \int_{2^k\delta}^{2^{k+1}\delta} \int_{\mathbb{R}^5} \frac{K_M(x,y)}{(M^2|t|)^{50}} \tilde{\chi}_k(t,y) [\tilde{P}_M |\nabla|^{\frac{1}{2}} F(u(t))](y)\, \mathrm{d}y\, \mathrm{d}t \Big\|_{L_x^2} \\
\lesssim_u (N^2\delta)^{-49} G\big(\frac{N}{16}\big).
\end{aligned}$$

Choosing N sufficiently large depending on δ, η, we get the desired result.

From (8.11), (8.13), Lemma 8.3, Lemma 8.4, Lemma 8.5, it concludes Proposition 8.1, which in turn proves Theorem 8.1.

9 No soliton

In this section we prove

Theorem 9.1. *There exists no non-zero soliton-like solution in the sense of Theorem 1.3.*

Proof. We argue by contradiction. Assume that there exists such a soliton solution, then by Theorem 6.1, Theorem 8.1, $u \in L_t^\infty H_x^s (s \geq 1)$, and u has the energy of the form

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^5} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{4}\iint_{\mathbb{R}^5\times\mathbb{R}^5} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y.$$

Now, define

$$M_a(t) := 2\mathrm{Im}\int_{\mathbb{R}^5} \bar{u}(t,x)\vec{a}(x)\cdot\nabla u(t,x)\,\mathrm{d}x,$$

where $a(x) = x\psi(\frac{|x|}{R})$, ψ is a smooth, radial function such that

$$\psi(r) = \begin{cases} 1, & r \leq 1 \\ 0, & r \geq 2. \end{cases}$$

Then, by the Cauchy-Schwarz inequality, we have

$$|M_a(t)| \leq R\|u\|_2\|\nabla u\|_2 \lesssim_u R. \tag{9.1}$$

We should prove by our assumption $\sup_{t\in\mathbb{R}} \big\||\nabla|^{\frac{1}{2}}u\big\|_2 < \frac{\sqrt{6}}{3}\big\||\nabla|^{\frac{1}{2}}Q\big\|_2$ that $M_a(t)$ is an increasing function of time, i.e., $\partial_t M_a(t) > 0$. Thus, a contradiction with (9.1)

A few computations with equation (1.1) yields

$$\begin{aligned}
\partial_t M_a(t) \;&=\; 12E(u(t)) - 2\int_{\mathbb{R}^5} |\nabla u|^2\,\mathrm{d}x && (9.2)\\
&\quad - \int_{\mathbb{R}^5}\Big[\frac{24}{R|x|}\psi'\big(\frac{|x|}{R}\big) + \frac{11}{R^2}\psi''\big(\frac{|x|}{R}\big) + \frac{|x|}{R^3}\psi'''\big(\frac{|x|}{R}\big)\Big]|u(t,x)|^2\,\mathrm{d}x && (9.3)\\
&\quad + 4\int_{\mathbb{R}^5}\Big[\psi\big(\frac{|x|}{R}\big) - 1 + \frac{|x|}{R}\psi'\big(\frac{|x|}{R}\big)\Big]|\nabla u(t,x)|^2\,\mathrm{d}x && (9.4)\\
-3\iint_{\mathbb{R}^5\times\mathbb{R}^5}&\Big[x\psi\big(\frac{|x|}{R}\big) - y\psi\big(\frac{|y|}{R}\big) - (x-y)\Big]\cdot\frac{x-y}{|x-y|^5}|u(t,x)|^2|u(t,y)|^2\,\mathrm{d}x\,\mathrm{d}y. && (9.5)
\end{aligned}$$

We will prove that (9.3), (9.4), (9.5) are sufficiently small compared to (9.2).

Note that (9.3) has a trivial bound R^{-2}.

Now, let $\eta > 0$ be a small number to be chosen later. From Lemma 8.1, there exists $R = R(\eta)$ such that for all $t \in \mathbb{R}$

$$\int_{|x| \geq \frac{R}{4}} |u(t,x)|^2 \,\mathrm{d}x \leq \eta. \tag{9.6}$$

Define χ as a smooth cutoff to the region $|x| \geq \frac{R}{2}$ with $\nabla\chi$ be bounded by R^{-1} and supported on $\{|x| \sim R\}$. Since $u \in C_t^0 H^s (s > 1)$, using the interpolation theorem and (9.6), we deduce

$$\begin{aligned} |(9.4)| \lesssim \|\chi \nabla u(t)\|_2^2 \lesssim \quad & \|\nabla(\chi u)\|_2^2 + \|u \nabla \chi\|_2^2 \lesssim \|\chi u(t)\|_2^{\frac{2(s-1)}{s}} \|u(t)\|_{H^s}^{\frac{2}{s}} + \eta \\ \lesssim_u & \ \eta^{\frac{s-1}{s}} + \eta. \end{aligned}$$

It remains to estimate (9.5). We divide the integration into three parts.

$$\begin{aligned} &(9.5) = \\ &\quad 2\mu \iint_{\substack{|x|\geq R \\ |y| \geq R}} \Big(x\Big(\psi(\frac{|x|}{R}) - 1\Big) - y\Big(\psi(\frac{|y|}{R}) - 1\Big)\Big) \cdot \frac{x-y}{|x-y|^5} |u(t,x)|^2 |u(t,y)|^2 \,\mathrm{d}x \,\mathrm{d}y \\ &\quad + 2\mu \iint_{\substack{|x|\geq R \\ |y| < R}} x\Big(\psi(\frac{|x|}{R}) - 1\Big) \cdot \frac{x-y}{|x-y|^5} |u(t,x)|^2 |u(t,y)|^2 \,\mathrm{d}x \,\mathrm{d}y \\ &\quad - 2\mu \iint_{\substack{|x|< R \\ |y| \geq R}} y\Big(\psi(\frac{|y|}{R}) - 1\Big) \cdot \frac{x-y}{|x-y|^5} |u(t,x)|^2 |u(t,y)|^2 \,\mathrm{d}x \,\mathrm{d}y \\ &\quad := I_1 + I_2 + I_3. \end{aligned}$$

We first estimate I_1. By the Gagliardo-Nirenberg inequality of convolution type and (9.6)

$$|I_1| \lesssim \iint_{\substack{|x|\geq R \\ |y| \geq R}} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^3} \,\mathrm{d}x \,\mathrm{d}y \lesssim \|\chi u\|_2 \|\nabla u\|_2^3 \lesssim_u \eta^{1/2}.$$

To estimate I_2, using the Hardy-Littlewood-Sobolev inequality, Lemma 3.1, Sobolev's embedding theorem,

$$\begin{aligned} |I_2| \quad \lesssim & \quad \iint_{\substack{|x|>2R \\ |y|<R}} |x| \frac{|u(x)|^2 |u(y)|^2}{|x-y|^4} \,\mathrm{d}x \,\mathrm{d}y \\ & \quad + \iint_{\substack{R<|x|\leq 2R \\ |y|<R}} \Big| x\Big(\psi(\frac{|x|}{R}) - 1\Big) \Big| \frac{|u(x)|^2 |u(y)|^2}{|x-y|^4} \,\mathrm{d}x \,\mathrm{d}y \\ \lesssim & \quad \iint_{\mathbb{R}^5 \times \mathbb{R}^5} \frac{|\chi u(x)|^2 |u(y)|^2}{|x-y|^3} \,\mathrm{d}x \,\mathrm{d}y + R^{-\frac{3}{4}} \iint_{\mathbb{R}^5 \times \mathbb{R}^5} \frac{|x|^{7/4} |u| \cdot |\chi u(x)| |u(y)|^2}{|x-y|^4} \,\mathrm{d}x \,\mathrm{d}y \\ \lesssim & \quad \|\chi u\|_2 \|\nabla(\chi u)\|_2 \|\nabla u\|_2^2 + R^{-\frac{3}{4}} \big\| |x|^{7/4} u \big\|_{L_x^\infty} \|\chi u\|_2 \|u\|_{H_x^1}^2 \\ \lesssim_u & \quad \eta^{\frac{2s-1}{2s}} + R^{-\frac{3}{4}} \eta^{\frac{1}{2}}. \end{aligned}$$

Note that in the last inequality, we used the interpolation as that to estimate (9.4).

I_3 can be estimated in the same argument.

Thus, choosing η sufficiently small depending on u, R sufficiently large depending on u and η, we have

$$|(9.3)| + |(9.4)| + |(9.5)| \lesssim \frac{1}{100} \times \Big[12E(u(t)) - 2\int_{\mathbb{R}^5} |\nabla u|^2 \,\mathrm{d}x\Big].$$

On the other hand, as $\sup_{t\in\mathbb{R}} \big\||\nabla|^{\frac{1}{2}}u\big\|_2 < \frac{\sqrt{6}}{3}\big\||\nabla|^{\frac{1}{2}}Q\big\|_2$, using the Hardy-Littlewood-Sobolev type inequality (3.1), we see $(9.2) > 0$. Hence $\partial_t M_a(t) > 0$. This concludes the proof of Theorem 9.1.

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